# Neutral stochastic differential equations driven by a fractional Brownian motion with impulsive effects and varying-time delays

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**Abstract**

In this paper we investigate the existence, uniqueness and asymptotic behaviors of mild solutions to neutral stochastic differential equations with delays and nonlinear impulsive effects, driven by fractional Brownian motion with the Hurst index $H > \frac{1}{2}$ in a Hilbert space. The cases of finite and infinite delays are discussed separately.

*Keywords:* Fractional Brownian motion, Mild solutions, Impulses, Delays, Asymptotic behaviors.
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## 1. Introduction

The fractional Brownian motion (fBm) is a family of centered Gaussian processes with continuous sample paths indexed by the Hurst parameter $H \in (0, 1)$. It is a self-similar process with stationary increments and has a long-memory when $H > \frac{1}{2}$. These significant properties make fractional Brownian motion a natural candidate as a model for noise in a wide variety of physical phenomena, such as mathematical finance, communication networks, hydrology and medicine. Therefore, it is important to study stochastic calculus with respect to fBm and related problems (we refer the reader to Mishura (2008) and the references therein for a more complete presentation of this subject).

Recently, stochastic delay differential equations driven by fBm have attracted a lot attentions of works. The first results are established by Ferrante and Rovira (2006). Since then,

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based on different settings, various forms of equations have been studied. For example, the case of finite-dimensional equations has been studied by Neuenkirch et al. (2008), Boufoussi and Hajji (2011), León and Tindel (2012), Besalú and Rovira (2012), Dung (2012) and the case of equations in a Hilbert space has been considered by Caraballo et al. (2011), Boufoussi and Hajji (2012) and Boufoussi et al. (2012). In most of the works, the delays are finite.

On the other hand, it is known that the impulsive effects exist widely in different areas of real world such as mechanics, electronics, telecommunications, neural networks, finance and economics, etc. (see, for instance, Lakshmikantham et al. (1989)). This is due to the fact that the states of many evolutionary processes are often subject to instantaneous perturbations and experience abrupt changes at certain moments of time. The duration of these changes is very short and negligible in comparison with the duration of the process considered, and can be thought as impulses. Hence, it is important to take into account the effect of impulses in the investigation of stochastic delay differential equations driven by fBm. However, to our best knowledge, no work has been reported in the present literature regarding the theory of stochastic differential equations driven by fBm with impulsive effects. The aim of this paper is to study one of such equations.

Our work is inspired by the one of Boufoussi and Hajji (2012) where the following neutral stochastic differential equation driven by fBm with finite delays has been studied

$$\begin{cases} d[x(t) + g(t, x(t - r(t)))] = \big[Ax(t) + f(t, x(t - \rho(t)))\big]dt + \sigma(t)dW^H(t),\ t \geq 0, \\ \\ x(t) = \phi(t),\ t \in [-\tau, 0]\ (0 < \tau < \infty). \end{cases}$$

In this paper, we are interested in the existence, uniqueness and asymptotic behaviors of mild solutions for a neutral stochastic differential equation with finite or infinite delays and impulsive effects of the following form in a Hilbert space

$$\begin{cases} d[x(t) + g(t, x(t - r(t)))] = \big[Ax(t) + f(t, x(t - \rho(t)))\big]dt + \sigma(t)dW^H(t),\ t \geq 0, t \neq t_k, \\ \\ \Delta x(t_k) := x(t_k^+) - x(t_k) = I_k(x(t_k)),\ \ k \in \mathbb{N}, \\ \\ x(t) = \phi(t),\ t \in (-\tau, 0]\ (0 < \tau \leq \infty), \end{cases} \tag{1.1}$$

where $A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space $X$ with norm $\|.\|$, $W^H$ is a fractional Brownian motion with $H > \frac{1}{2}$ on a real and separable Hilbert space $Y$, $r, \rho : [0, \infty) \to [0, \tau)$ are continuous, $\mathbb{N}$ denotes the set of positive integers, the impulsive moments satisfy $0 < t_1 < t_2 < .....,\ \lim_{k\to\infty} t_k = \infty$, and $f, g : [0, \infty) \times X \to X$, $\sigma : [0, \infty) \to \mathcal{L}_2^0(Y, X)$, $I_k : X \to X$ are defined later, the initial data $\phi \in C((-\tau, 0], X)$ the space of all continuous functions from $(-\tau, 0]$ to $X$ and has finite second moments. The space $\mathcal{L}_2^0(Y, X)$ will be defined in Section 2.

It is known that a fBm is neither a semimartingale nor a Markov process. Hence, the traditional tools of Itô stochastic analysis can not be applied effectively in studying the solution of equations driven by fBm. Because of those reasons, even in the case of equations without impulses, the asymptotic behaviors of solutions have only been investigated by a few authors (see e.g. Duncan et al. (2005); Caraballo et al. (2011); Boufoussi and Hajji (2012)). Furthermore, since the appearance of impulses in the equation (1.1), we need to find the new techniques which are different from that used by Boufoussi and Hajji (2012) to investigate the asymptotic behaviors of solutions of (1.1). The main tool of this paper is the fixed point theory which was proposed by Burton (2006).

The rest of this paper is organized as follows. In Section 2, we briefly present some basic notations and preliminaries. Section 3 is devoted to study the existence, uniqueness and asymptotic behaviors of mild solutions; an example is also provided in this section. The conclusion is given in Section 4.

## 2. Preliminaries

We first recall the definition of Wiener integrals with respect to an infinite dimensional fractional Brownian motion with Hurst index $H > \frac{1}{2}$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $T > 0$ be an arbitrary fixed horizon. An one-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $\beta^H = \{\beta^H(t), 0 \le t \le T\}$ with the covariance function $R(t,s) = E[\beta^H(t)\beta^H(s)]$

$$R(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

It is known that $\beta^H(t)$ with $H > \frac{1}{2}$ admits the following Volterra representation

$$\beta^H(t) = \int_0^t K(t,s)d\beta(s), \tag{2.1}$$

where $\beta$ is a standard Brownian motion and the Volterra kernel $K(t,s)$ is given by

$$K(t,s) = c_H \int_s^t (u-s)^{H-\frac{3}{2}} \left(\frac{u}{s}\right)^{H-\frac{1}{2}} du, \ t \ge s.$$

For the deterministic function $\varphi \in L^2([0,T])$, the fractional Wiener integral of $\varphi$ with respect to $\beta^H$ is defined by

$$\int_0^T \varphi(s)d\beta^H(s) = \int_0^T K_H^*\varphi(s)d\beta(s),$$

where $K_H^*\varphi(s) = \int_s^T \varphi(r)\frac{\partial K}{\partial r}(r,s)dr$.

Let $X$ and $Y$ be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operators from $Y$ to $X$. For the sake of convenience, we shall use the same notation to denote the norms in $X, Y$ and $\mathcal{L}(Y,X)$. Let $\{e_n, n = 1,2,...\}$ be a complete orthonormal basis in $Y$ and $Q \in \mathcal{L}(Y,X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n, n = 1,2,...$ are non-negative real numbers. We define the infinite dimensional fBm on $Y$ with covariance $Q$ as

$$W^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where $\beta_n^H(t)$ are real, independent fBm's. This process is a $Y$-valued Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle W^H(t),x\rangle\langle W^H(s),y\rangle = R(t,s)\langle Q(x),y\rangle \text{ for all } x,y \in Y \text{ and } t,s \in [0,T].$$

In order to define Wiener integrals with respect to the $Q$-fBm $W^H(t)$, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ of all $Q$-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a $Q$-Hilbert-Schmidt operator if

$$\|\psi\|_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

The fractional Wiener integral of the function $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ with respect to $Q$-fBm is defined by

$$\int_0^t \psi(s)dW^H(s) = \sum_{n=1}^{\infty}\int_0^t \sqrt{\lambda_n}\psi(s)e_n d\beta_n^H(s) = \sum_{n=1}^{\infty}\int_0^t \sqrt{\lambda_n}K_H^*(\psi e_n)(s)d\beta_n(s), \tag{2.2}$$

where $\beta_n$ is the standard Brownian motion used to present $\beta_n^H$ as in (2.1).

We have the following fundamental inequality which was proved in Boufoussi and Hajji (2012).

**Lemma 2.1.** *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int\limits_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$ then the above sum in (2.2) is well defined as an $X$-valued random variable and we have*

$$E\Big\| \int\limits_0^t \psi(s)dW^H(s)\Big\|^2 \leq 2Ht^{2H-1}\int\limits_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

We now suppose that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of $A$, and the semigroup $S(t)$ is uniformly bounded. That is to say, $\|S(t)\| \leq M$ for some constant $M \geq 1$ and every $t \geq 0$. Then, for $0 < \alpha \leq 1$, it is possible to define the fractional power operator $(-A)^\alpha$ as a closed linear operator on its domain $\mathcal{D}(-A)^\alpha$. Furthermore, the subspace $\mathcal{D}(-A)^\alpha$ is dense in $X$ and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \;\; x \in \mathcal{D}(-A)^\alpha$$

defines a norm on $X_\alpha := \mathcal{D}(-A)^\alpha$. The following properties are well known (see, Pazy (1983)).

**Lemma 2.2.** *Under the above conditions, the following properties hold:*
*(i) $X_\alpha$ is a Banach space for $0 < \alpha \leq 1$,*
*(ii) If the resolvent operator of $A$ is compact, then the embedding $X_\beta \subset X_\alpha$ is continuous and compact for $0 < \alpha \leq \beta$,*
*(iii) For every $0 < \alpha \leq 1$, there exists $M_\alpha$ such that*

$$\|(-A)^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\lambda t}, \;\; \lambda > 0, t \geq 0.$$

## 3. The main result

In this section, we first establish the results for the case of finite delays. Then, the case of infinite delays can be proved similarly. Our method is based on the contraction mapping principle.

Let $0 < \tau \leq \infty$, we have the following definition of mild solutions for Eq. (1.1).

**Definition 3.1.** An $X$-valued stochastic process $\{x(t), t \in (-\tau, \infty)\}$ is called a mild solution of Eq. (1.1) if $x(t) = \phi(t)$ on $(-\tau, 0]$, and the following conditions hold:

(i) $x(.)$ is continuous on $(0, t_1]$ and each interval $(t_k, t_{k+1}], k \in \mathbb{N}$,

(ii) for each $t_k$, $x(t_k^+) = \lim_{t \to t_k^+} x(t)$ exists,

(iii) for each $t \geq 0$, we have $a.s.$

$$x(t) = S(t)(\phi(0) + g(0, \phi(-r(0)))) - g(t, x(t - r(t))) - \int_0^t AS(t-s)g(s, x(s - r(s)))ds$$
$$+ \int_0^t S(t-s)f(s, x(s - \rho(s)))ds + \int_0^t S(t-s)\sigma(s)dW^H(s)$$
$$+ \sum_{0<t_k<t} S(t - t_k)I_k(x(t_k)). \quad (3.1)$$

In order to prove the required results, we assume the following conditions:

$(H_1)$ $A$ is the infinitesimal generator of an analytic semigroup, $(S(t))_{t\geq 0}$, of bounded linear operators on $X$. Moreover, $S(t)$ satisfies the condition that there exist positive constants $M, \lambda$ such that

$$\|S(t)\| \leq Me^{-\lambda t}, \;\; t \geq 0.$$

$(H_2)$ There exists $L_1 > 0$ such that, for all $t \geq 0$, $x, y \in X$

$$\|f(t, x) - f(t, y)\|^2 \leq L_1 \|x - y\|^2.$$

$(H_3)$ There exist constants $0 < \beta < 1, L_2 > 0$ such that the function $g$ is $X_\beta$-valued and satisfies for all $t \geq 0$, $x, y \in X$

$$\|(-A)^\beta g(t, x) - (-A)^\beta g(t, y)\|^2 \leq L_2 \|x - y\|^2.$$

$(H_4)$ The function $(-A)^\beta g$ is continuous in the quadratic mean sense:

$$\text{For all functions } x, \quad \lim_{t \to s} E\|(-A)^\beta g(t, x(t)) - (-A)^\beta g(s, x(s))\|^2 = 0.$$

$(H_5)$ There exist some positive numbers $q_k, k \in \mathbb{N}$ such that

$$\|I_k(x) - I_k(y)\| \leq q_k \|x - y\|$$

for all $x, y \in X$ and $\sum\limits_{k=1}^{\infty} q_k < \infty$.

$(H_6)$ The function $\sigma : [0, \infty) \to \mathcal{L}_2^0(Y, X)$ satisfies

$$\int_0^{\infty} e^{2\gamma s} \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty \text{ for some } \gamma > 0.$$

We first consider the case of finite delays, i.e., $\tau < \infty$.

**Theorem 3.1.** *(finite delays). Assume that $f(t, 0) = g(t, 0) = I_k(0) = 0, \ \forall\, t \geq 0, \ k \in \mathbb{N}$, the assumptions $(H_1)$-$(H_5)$ hold and that*

$$4\Bigg( L_2\|(-A)^{-\beta}\|^2 + M_{1-\beta}^2 L_2 \Gamma^2(\beta)\lambda^{-2\beta} + M^2 L_1 \lambda^{-2} + M^2 \big(\sum_{k=1}^{\infty} q_k\big)^2 \Bigg) < 1, \tag{3.2}$$

*where $\Gamma(.)$ is the Gamma function, $M_{1-\beta}$ is the corresponding constant in Lemma 2.2. Then the mild solution to (1.1) exists uniquely and is exponential decay to zero in mean square, i.e., there exists a pair of positive constants $a > 0$ and $M^* = M^*(\phi, a)$ such that*

$$E\|x(t)\|^2 \leq M^* e^{-at}, \ \forall\, t \geq 0.$$

*Proof.* Denote by $\mathcal{S}$ the space of all stochastic processes $x(t, \omega) : (-\tau, \infty) \times \Omega \to X$ satisfying $x(t) = \phi(t), t \in (-\tau, 0]$ and the conditions (i), (ii) in Definition 3.1 and there exist some constants $a > 0$ and $M^* = M^*(\phi, a) > 0$ such that

$$E\|x(t)\|^2 \leq M^* e^{-at}, \ t \geq 0. \tag{3.3}$$

It is routine to check that $\mathcal{S}$ is a Banach space endowed with a norm $|x|^2_{\mathcal{S}} = \sup\limits_{t \geq 0} E|x(t)|^2$.

Without loss of generality, we may assume that $a < \lambda$. We define the operator $\Phi$ on $\mathcal{S}$ by $(\Phi x)(t) = \phi(t), t \in (-\tau, 0]$ and

$$(\Phi x)(t) = S(t)(\phi(0) + g(0, \phi(-r(0)))) - g(t, x(t - r(t))) - \int_0^t AS(t-s) g(s, x(s - r(s))) ds$$
$$+ \int_0^t S(t-s) f(s, x(s - \rho(s))) ds + \int_0^t S(t-s)\sigma(s) dW^H(s)$$
$$+ \sum_{0 < t_k < t} S(t - t_k) I_k(x(t_k)) := \sum_{i=1}^{6} P_i(t), \ t \geq 0.$$

To get desired results, it is enough to show that the operator $\Phi$ has a unique fixed point in $\mathcal{S}$. For this purpose, we use the contraction mapping principle.

*Step 1.* We first verify that $\Phi(\mathcal{S}) \subset \mathcal{S}$. For convenience of notation, we denote by $M_i^*, i = 1, 2, ...$ the finite positive constants depending on $\phi, a$. By the assumption $(H_1)$ we have

$$E\|P_1(t)\|^2 \leq M^2 E\|\phi(0) + g(0, \phi(-r(0)))\|^2 e^{-\lambda t} \leq M_1^* e^{-\lambda t}. \tag{3.4}$$

To estimate $P_i(t), i = 2, ..., 6$, we observe that for $x \in \mathcal{S}$ and $u(t) = r(t)$ or $\rho(t)$, the following useful estimate holds

$$\begin{aligned} E\|x(t-u(t))\|^2 &\leq (M^* e^{-a(t-u(t))} + E\|\phi(t-u(t))\|^2) \\ &\leq (M^* e^{-a(t-u(t))} + E\|\phi\|_C^2 e^{-a(t-u(t))}) \\ &\leq (M^* + E\|\phi\|_C^2) e^{a\tau} e^{-at}, \end{aligned}$$

where $\|\phi\|_C = \sup\limits_{-\tau < s \leq 0} \|\phi(s)\| < \infty$. Then by assumption $(H_3)$ we have

$$\begin{aligned} E\|P_2(t)\|^2 &\leq \|(-A)^{-\beta}\|^2 E\|(-A)^{\beta} g(t, x(t-r(t))) - (-A)^{\beta} g(t, 0)\|^2 \\ &\leq L_2 \|(-A)^{-\beta}\|^2 E\|x(t-r(t))\|^2 \\ &\leq L_2 \|(-A)^{-\beta}\|^2 (M^* + E\|\phi\|_C^2) e^{a\tau} e^{-at} \\ &\leq M_2^* e^{-at}. \end{aligned} \tag{3.5}$$

.

Using Lemma 2.2, Hölder's inequality and assumption $(H_3)$ we obtain that

$$\begin{aligned} E\|P_3(t)\|^2 &= E\Big\|\int_0^t AS(t-s) g(s, x(s-r(s))) ds\Big\|^2 \\ &\leq \int_0^t \|(-A)^{1-\beta} S(t-s)\| ds \int_0^t \|(-A)^{1-\beta} S(t-s)\| E\|(-A)^{\beta} g(s, x(s-r(s)))\|^2 ds \\ &\leq M_{1-\beta}^2 L_2 \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} E\|x(s-r(s))\|^2 ds \\ &\leq M_{1-\beta}^2 L_2 \tfrac{\Gamma(\beta)}{\lambda^{\beta}} \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} (M^* + E\|\phi\|_C^2) e^{a\tau} e^{-as} ds \\ &\leq M_{1-\beta}^2 L_2 \tfrac{\Gamma(\beta)}{\lambda^{\beta}} (M^* + E\|\phi\|_C^2) e^{a\tau} e^{-at} \int_0^t (t-s)^{\beta-1} e^{(a-\lambda)(t-s)} ds \\ &\leq M_{1-\beta}^2 L_2 \tfrac{\Gamma^2(\beta)}{\lambda^{\beta}(\lambda-a)^{\beta}} (M^* + E\|\phi\|_C^2) e^{a\tau} e^{-at}. \end{aligned}$$

We therefore have

$$E\|P_3(t)\|^2 \leq M_3^* e^{-at}. \tag{3.6}$$

Similarly, we obtain by assumption $(H_2)$ that

$$\begin{aligned} E\|P_4(t)\|^2 &= E\Big\|\int_0^t S(t-s)f(s,x(s-\rho(s)))ds\Big\|^2 \\ &\leq M^2 L_1 \int_0^t e^{-\lambda(t-s)}ds \int_0^t e^{-\lambda(t-s)} E\|x(s-\rho(s))\|^2 ds \\ &\leq M^2 L_1 \lambda^{-1} \int_0^t e^{-\lambda(t-s)}(M^* + E\|\phi\|_C^2)e^{a\tau}e^{-as}ds \\ &\leq M^2 L_1 \lambda^{-1}(\lambda-a)^{-1}(M^* + E\|\phi\|_C^2)e^{a\tau}e^{-at} \\ &\leq M_4^* e^{-at}. \end{aligned} \tag{3.7}$$

By using Lemma 2.1 we get that

$$E\|P_5(t)\|^2 \leq 2M^2 H t^{2H-1} \int_0^t e^{-2\lambda(t-s)} \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds.$$

From this inequality we can infer that

$$E\|P_5(t)\|^2 \leq 2M^2 H t^{2H-1} e^{-2\lambda' t} \int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds, \tag{3.8}$$

where $\lambda' = \lambda \wedge \gamma$. Indeed, if $\lambda < \gamma$, then $\lambda' = \lambda$ and we have

$$E\|P_5(t)\|^2 \leq 2M^2 H t^{2H-1} e^{-2\lambda t} \int_0^t e^{2\lambda s} \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds \leq 2M^2 H t^{2H-1} e^{-2\lambda' t} \int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds.$$

If $\gamma < \lambda$, then $\lambda' = \gamma$ and we have

$$\begin{aligned} E\|P_5(t)\|^2 &\leq 2M^2 H t^{2H-1} e^{-2\gamma t} \int_0^t e^{-2(\lambda-\gamma)(t-s)} e^{2\gamma s} \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds \\ &\leq 2M^2 H t^{2H-1} e^{-2\lambda' t} \int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds. \end{aligned}$$

Since $\sup_{t\geq 0}(t^{2H-1}e^{-\lambda' t}) < \infty$, this, together with (3.8), gives us

$$E\|P_5(t)\|^2 \leq M_5^* e^{-\lambda' t}. \tag{3.9}$$

From $(H_5)$ and Hölder inequality, we get the following estimate for $P_6(t)$

$$
\begin{aligned}
E\|P_6(t)\|^2 &= E\Big\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k))\Big\|^2 \\
&\le E\Big(\sum_{0<t_k<t}\|S(t-t_k)\|\ \|I_k(x(t_k))-I_k(0)\|\Big)^2 \\
&\le M^2 E\Big(\sum_{0<t_k<t} e^{-\lambda(t-t_k)} q_k\|x(t_k)\|\Big)^2 \\
&\le M^2 \sum_{0<t_k<t} q_k \sum_{0<t_k<t} q_k e^{-2\lambda(t-t_k)} E\|x(t_k)\|^2 \\
&\le M^2 \sum_{k=1}^{\infty} q_k \sum_{0<t_k<t} q_k e^{-2\lambda(t-t_k)} M^* e^{-at_k} \\
&\le M^2 M^* e^{-at} \sum_{k=1}^{\infty} q_k \sum_{0<t_k<t} q_k e^{(a-2\lambda)(t-t_k)} \\
&\le M^2 M^* e^{-at}\Big(\sum_{k=1}^{\infty} q_k\Big)^2 \le M_6^* e^{-at}.
\end{aligned}
\tag{3.10}
$$

Combining (3.4)-(3.7), (3.9) and (3.10) we see that there exist $\overline{M}^* > 0$ and $\overline{a} > 0$ such that

$$E\|(\Phi x)(t)\|^2 \le \overline{M}^* e^{-\overline{a}t},\ t \ge 0.$$

Moreover, it is easy to check that $(\Phi x)(t)$ satisfies the conditions (i), (ii) in Definition 3.1. Hence, we can conclude that $\Phi(\mathcal{S}) \subset \mathcal{S}$.

*Step 2.* We now show that $\Phi$ is a contraction mapping. For any $x, y \in \mathcal{S}$, we have

$$E\|(\Phi x)(t) - (\Phi y)(t)\|^2 \le 4\sum_{i=1}^{4} Q_i.$$

Since $x(t) = y(t) = \phi(t), t \in (-\tau, 0]$, this implies that

$$E\|x(t-r(t)) - y(t-r(t))\|^2 \le \sup_{t\ge 0} E\|x(t)-y(t)\|^2.$$

Then we have by assumption $(H_3)$

$$
\begin{aligned}
Q_1 &= E\|g(t, x(t-r(t))) - g(t, y(t-r(t)))\|^2 \\
&\le L_2\|(-A)^{-\beta}\|^2 E\|x(t-r(t)) - y(t-r(t))\|^2 \\
&\le L_2\|(-A)^{-\beta}\|^2 \sup_{t\ge 0} E\|x(t)-y(t)\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
Q_2 &= E\Big\|\int_0^t AS(t-s)[g(s,x(s-r(s)))-g(s,y(s-r(s)))]ds\Big\|^2\\
&\le M_{1-\beta}^2 L_2\int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}ds\int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}E\|x(s-r(s))-y(s-r(s))\|^2 ds\\
&\le M_{1-\beta}^2 L_2\frac{\Gamma(\beta)}{\lambda^{\beta}}\int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}E\|x(s-r(s))-y(s-r(s))\|^2 ds\\
&\le M_{1-\beta}^2 L_2\frac{\Gamma^2(\beta)}{\lambda^{2\beta}}\sup_{t\ge 0}E\|x(t)-y(t)\|^2.
\end{aligned}
$$

By assumption $(H_2)$

$$
\begin{aligned}
Q_3 &= E\Big\|\int_0^t S(t-s)[f(s,x(s-\rho(s)))-f(s,y(s-\rho(s)))]ds\Big\|^2\\
&\le M^2L_1\int_0^t e^{-\lambda(t-s)}ds\int_0^t e^{-\lambda(t-s)}E\|x(s-\rho(s))-y(s-\rho(s))\|^2ds\\
&\le M^2L_1\lambda^{-1}\int_0^t e^{-\lambda(t-s)}E\|x(s-\rho(s))-y(s-\rho(s))\|^2ds\\
&\le M^2L_1\lambda^{-2}\sup_{t\ge 0}E\|x(t)-y(t)\|^2.
\end{aligned}
$$

By assumption $(H_5)$

$$
\begin{aligned}
Q_4 &= E\Big\|\sum_{0<t_k<t}S(t-t_k)[I_k(x(t_k))-I_k(y(t_k))]\Big\|^2\\
&\le M^2\Big(\sum_{0<t_k<t}e^{-\lambda(t-t_k)}q_kE\|x(t_k)-y(t_k)\|\Big)^2\\
&\le M^2\Big(\sum_{k=1}^{\infty}q_k\Big)^2\sup_{t\ge 0}E\|x(t)-y(t)\|^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
E\|(\Phi x)(t)-(\Phi y)(t)\|^2 \le 4\Bigg(&L_2\|(-A)^{-\beta}\|^2+M_{1-\beta}^2L_2\Gamma^2(\beta)\lambda^{-2\beta}+M^2L_1\lambda^{-2}\\
&+M^2\Big(\sum_{k=1}^{\infty}q_k\Big)^2\Bigg)\sup_{t\ge 0}E\|x(t)-y(t)\|^2.
\end{aligned}
$$

By the condition (3.2), we claim that $\Phi$ is contractive. So, applying the Banach fixed point principle, the proof is complete. □

*Remark* 3.1. The assumption $f(t,0)=g(t,0)=I_k(0)=0,\ \forall\ t\ge 0,\ k\in\mathbb{N}$ is very popular in the context of stability problems. Theorem 3.1 remains true if we replace the conditions

$f(t,0) = g(t,0) = 0$, $\forall\, t \geq 0$, by the conditions below which used by Boufoussi and Hajji (2012) to prove the exponential decay of mild solutions

$$\|f(t,x)\|^2 + \|(-A)^\beta g(t,x)\|^2 \leq L_3\|x\|^2 + e^{-bt},$$

where $L_3, b$ are some finite positive constants. Thus our results as compared with the one of Boufoussi and Hajji (2012) show that under some suitable assumptions, the appearance of impulses does not affect the existence, uniqueness and exponential decay of mild solutions.

*Remark* 3.2. When $g = 0$ and $I_k = 0, k \in \mathbb{N}$, our equation (1.1) reduces to the following equation

$$\begin{cases} dx(t) = \big[Ax(t) + f(t, x(t-\rho(t)))\big]dt + \sigma(t)dW^H(t),\ t \geq 0, \\ \\ x(t) = \phi(t),\ t \in (-\tau, 0]. \end{cases} \tag{3.11}$$

This equation has been recently investigated by Caraballo et al. (2011). In order to get similar results to our ones, they assumed that $\rho(t)$ is differentiable and satisfies $|\frac{1}{1-\rho'(t)}| < \rho^*$, where $\rho^*$ is a positive constant. Thus our results improve the one of Caraballo et al. (2011).

We now turn our attention to the case of infinite delays ($\tau = \infty$). We assume that

$$t - r(t) \to \infty \text{ and } t - \rho(t) \to \infty \text{ as } t \to \infty.$$

**Theorem 3.2.** *(infinite delays). Under the conditions of Theorem 3.1, the mild solution to (1.1) exists uniquely and converges to zero in mean square, i.e.,*

$$\lim_{t\to\infty} E\|x(t)\|^2 = 0.$$

*Proof.* Denote by $\mathcal{S}'$ the space of all stochastic processes $x(t,\omega) : (-\infty,\infty)\times\Omega \to X$ satisfying $x(t) = \phi(t), t \in (-\infty, 0]$ and the conditions (i), (ii) in Definition 3.1 and

$$\lim_{t\to\infty} E\|x(t)\|^2 = 0. \tag{3.12}$$

We define the operator $\Psi$ on $\mathcal{S}'$ by $(\Psi x)(t) = \phi(t), t \in (-\infty, 0]$ and

$$\begin{aligned}(\Psi x)(t) = S(t)(\phi(0) + g(0, \phi(-r(0)))) - g(t, x(t-r(t))) - \int_0^t AS(t-s)g(s, x(s-r(s)))ds \\ + \int_0^t S(t-s)f(s, x(s-\rho(s)))ds + \int_0^t S(t-s)\sigma(s)dW^H(s) \\ + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)) := \sum_{i=1}^{6} P_i(t),\ t \geq 0.\end{aligned}$$

Since $(\Psi x)(t) = (\Phi x)(t)$ on $[0, \infty)$, this implies that $\Psi$ is contractive. Hence, it remains to check $\Psi(\mathcal{S}') \subset \mathcal{S}'$. In order to obtain this claim, we need to show that $\lim\limits_{t\to\infty} E\|(\Psi x)(t)\|^2 = 0$ for all $x \in \mathcal{S}'$.

By the definition of $\mathcal{S}'$, assumption $(H_6)$ and the fact $t - r(t) \to \infty, t \to \infty$, we get

$$\lim_{t\to\infty} E\|P_1(t)\|^2 = \lim_{t\to\infty} E\|P_2(t)\|^2 = \lim_{t\to\infty} E\|P_5(t)\|^2 = 0.$$

We further have

$$\begin{aligned} E\|P_3(t)\|^2 &\le E\Big\|\int_0^t AS(t-s)g(s, x(s-r(s)))ds\Big\|^2 \\ &\le M_{1-\beta}^2 L_2 \int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}ds \int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}E\|x(s-r(s))\|^2 ds \\ &\le M_{1-\beta}^2 L_2 \Gamma(\beta)\lambda^{-\beta} \int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}E\|x(s-r(s))\|^2 ds. \end{aligned}$$

For any $x \in \mathcal{S}'$ and $\varepsilon > 0$ it follows from (3.12) that there exists $s_1 > 0$ such that $E\|x(s - r(s))\|^2 < \varepsilon$ for all $s \ge s_1$. Thus we obtain

$$E\|P_3(t)\|^2 \le M_{1-\beta}^2 L_2 \Gamma(\beta)\lambda^{-\beta} \int_0^{s_1} (t-s)^{\beta-1}e^{-\lambda(t-s)}E\|x(s-r(s))\|^2 ds + M_{1-\beta}^2 L_2 \Gamma^2(\beta)\lambda^{-2\beta}\varepsilon,$$

which proves that

$$\lim_{t\to\infty} E\|P_3(t)\|^2 \le M_{1-\beta}^2 L_2 \Gamma^2(\beta)\lambda^{-2\beta}\varepsilon, \ \forall\, \varepsilon > 0,$$

and hence, $\lim\limits_{t\to\infty} E\|P_3(t)\|^2 = 0$. In the same way we also have $\lim\limits_{t\to\infty} E\|P_4(t)\|^2 = 0$. Furthermore, since

$$\begin{aligned} E\|P_6(t)\|^2 &= E\Big\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k))\Big\|^2 \\ &\le M^2 \sum_{0<t_k<t} q_k \sum_{0<t_k<t} q_k e^{-2\lambda(t-t_k)}E\|x(t_k)\|^2 \\ &\le M^2 \sum_{k=1}^{\infty} q_k \sum_{0<t_k<s_1} q_k e^{-2\lambda(t-t_k)}E\|x(t_k)\|^2 + M^2\Big(\sum_{k=1}^{\infty} q_k\Big)^2\varepsilon, \end{aligned}$$

we can get that $\lim\limits_{t\to\infty} E\|P_6(t)\|^2 = 0$.

Once again, by applying the Banach fixed point principle we complete the proof of the theorem. $\square$

To illustrate the obtained theory, let us end this section with an example.

**Example.** Consider the following neutral stochastic partial differential equation with delays and impulsive effects

$$\begin{cases} \frac{\partial}{\partial t}[u(t,z)+\alpha_1 u(\sin t,z)] = \frac{\partial^2}{\partial z^2}u(t,z) \\ \qquad\qquad\qquad\qquad +\alpha_2 \sin u(\frac{t}{2},z) + e^{-t}\frac{dW^H(t)}{dt},\ t\geq 0, t\neq t_k, 0\leq z\leq \pi, \\ \Delta u(t_k,z) := u(t_k^+,z)-u(t_k,z) = \frac{\alpha_3}{2^k}u(t_k,z),\ \ k\in\mathbb{N}, \\ u(t,0)=u(t,\pi)=0, \\ u(t,z)=\phi(t,z),\ t\in(-\infty,0], 0\leq z\leq\pi, \end{cases} \tag{3.13}$$

where $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 \geq 0$ are constants.

Let $X = L^2([0,\pi])$ with the norm $\|.\|$ and inner product $\langle .,. \rangle$. Define $A : X \to X$ by $Ax = x''$ with domain

$$\mathcal{D}(A) := \{x\in X : x, x' \text{ are absolutely continuous } x''\in X, x(0)=x(\pi)=0\}.$$

Then, the equation (3.13) can be written in the form of the equations (1.1) with the coefficients $g(t,x)=\alpha_1 x,\ f(t,x)=\alpha_2\sin x,\ \sigma(t)=e^{-t}$, the delays $r(t)=t-\sin t,\ \rho(t)=\frac{t}{2}$, and the impulsive functions $I_k(x)=\frac{\alpha_3}{2^k}x, k\in\mathbb{N}$.

For the operator $A$, it is known from Pazy (1983) that the following properties hold:

- $Ax = \sum\limits_{n=1}^{\infty} n^2\langle x, e_n\rangle e_n,\ x\in\mathcal{D}(A)$, where $e_n(t)=\sqrt{\frac{2}{\pi}}\sin(nt), n=1,2,....$ is the orthogonal set of eigenvectors.
- $A$ is the infinitesimal generator of an analytic semigroup $(S(t))_{t\geq 0}$ in $X$ :

  $$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, e_n\rangle e_n, \text{ for all } x\in X \text{ and every } t>0.$$

  Furthermore, $\|S(t)\| \leq e^{-\pi^2 t},\ t\geq 0$.
- The bounded linear operator $(-A)^{\frac{3}{4}}$ is well defined and given by

  $$(-A)^{\frac{3}{4}}x = \sum_{n=1}^{\infty} n^{\frac{3}{2}}\langle x, e_n\rangle e_n$$

with domain $\mathcal{D}((-A)^{\frac{3}{4}}) := \{x \in X : \sum\limits_{n=1}^{\infty} n^{\frac{3}{2}} \langle x, e_n\rangle e_n \in X\}$. Furthermore, $\|(-A)^{\frac{3}{4}}\| = 1$ and $\|(-A)^{-\frac{3}{4}}\| \le \frac{1}{\Gamma(\frac{3}{4})} \int\limits_0^{\infty} t^{-\frac{1}{4}} \|S(t)\| dt \le \frac{1}{\pi^{\frac{3}{2}}}$.

Thus $(H_1)$ holds with $M = 1, \lambda = \pi^2$, $(H_2)$ holds with $L_1 = \alpha_2^2$, $(H_3)$ and $(H_4)$ hold with $\beta = \frac{3}{4}, L_2 = \|(-A)^{\frac{3}{4}}\|^2 \alpha_1^2 = \alpha_1^2$, $(H_5)$ holds with $q_k = \frac{\alpha_3}{2^k},\ k \in \mathbb{N}$ and $(H_6)$ holds with $\gamma = \frac{1}{2}$. Consequently, we can conclude, by Theorem 3.2, that the stochastic partial equation (3.13) has a unique mild solution and that this solution converges to zero in mean square if the parameters $\alpha_1, \alpha_2$ and $\alpha_3$ satisfy the following relation:

$$\frac{\alpha_1^2}{\pi^3} + M_{\frac{1}{4}}^2 \Gamma^2\left(\frac{3}{4}\right) \frac{\alpha_1^2}{\pi^3} + \frac{\alpha_2^2}{\pi^4} + \alpha_3^2 < \frac{1}{4}.$$

## 4. Conclusion

In this paper, we made a first attempt to study stochastic delay differential equations driven by fBm with impulsive effects. Our work extends the work of Boufoussi and Hajji (2012) and improves some results of Caraballo et al. (2011). In addition, we also discuss the case of infinite delays which has not yet been discussed in the context of stochastic delay differential equations driven by fBm.

A common feature of our paper and the papers by Boufoussi and Hajji (2012); Caraballo et al. (2011) is that only equations with additive noise are considered. It is natural to ask what we can say about equations with multiplicative noise. Let us consider the simplest case, that is, the case of scalar linear equations

$$dx(t) = ax(t)dt + \sigma x(t) dW^H(t),\ x(0) = x_0,$$

where $a, \sigma \neq 0$ and $x_0$ are real numbers. This equation admits a unique solution $x(t) = x_0 e^{at - \frac{\sigma^2}{2} t^{2H} + \sigma W^H(t)}$. Since $W^H(t)$ is a centered Gaussian process with the variance of $t^{2H}$, we can easily compute the second moment of $x(t)$ which is given by

$$E|x(t)|^2 = x_0^2 e^{2at + \sigma^2 t^{2H}},\ t \ge 0.$$

The above expression points out that $x(t)$ cannot converge to zero (and hence, cannot exponentially decays) in mean square for all $a$ and $\sigma$.

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