The existence of a positive solution for a generalized delay logistic equation with multifractional noise

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Abstract

The aim of this work is to prove the existence of a positive solution for a class of generalized delay logistic equations with multifractional noise. To do this, a method of semimartingale approximation is introduced.

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1. Introduction

The classical logistic equation

$$dN(t) = (\lambda N(t) - c[N(t)]^2)dt, \tag{1.1}$$

was proposed by Verhulst (1838) to describe population growth in a limited environment and since then it has remained very popular in population dynamics. A generalization of the logistic equations has been recently introduced by Yukalov et al. (2009) that reads

$$dN(t) = \left(\lambda N(t) - \frac{c[N(t)]^2}{a + bN(t-r)}\right)dt, \tag{1.2}$$

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where λ, a, b, c are positive constants. In the context of population dynamics, r characterizes the reaction time of the population to environmental constraints, a is the original carrying capacity and b is interpreted as the factor that controls the current carrying capacity as a proportion of the historical development. The term λ can be interpreted as the net birth rate with respect to the death and the term c represents the net competition versus the cooperation.

It is clear that the models (1.1) and (1.2) will be more realistic if noise is added. In fact, many stochastic versions of (1.1) with Brownian noise have been investigated. For example, the model

$$dN(t) = (\lambda N(t) - c[N(t)]^2)dt + \sigma N(t)dW(t),$$

is well-known as the stochastic logistic model or the so-called Verhulst model in population study.

In this paper, we use the multifractional Gaussian process as a noise and choose to make the net birth rate, λ, as a stochastic parameter then we could model $\gamma(\omega)$ by

$$\gamma(\omega) = \gamma + \sigma \text{ "multifractional noise"}, \tag{1.3}$$

where $\gamma = E[\gamma(\omega)], \sigma$ are deterministic. Replacing the delay term $\frac{c}{a+bN(t-r)}$ by a general function $g(t, N(t-r))$ and inserting (1.3) into Eq. (1.2), we get a stochastic version of the generalized delay logistic equations which has the form

$$\begin{aligned} &dN(t) = \big(\lambda N(t) - g(t, N(t-r))[N(t)]^2\big)dt + \sigma N(t)dB_h(t)\ ,\ t \in [0, T], \\ &N(t) = \phi(t)\,,\ t \in [-r, 0], \text{ where } \phi \in C[-r, 0]; \end{aligned} \tag{1.4}$$

where λ, σ are real constants and $B_h(t)$ is a Volterra-type multifractional Gaussian process.

Let $h : [0, +\infty) \to [a, b] \subset (1/2, 1)$ be a Hölder function of exponent $\beta > 0$, i.e. for any $t_1, t_2 \in [0, +\infty)$ such that $|t_1 - t_2| < 1$, there exists a constant $c_0 > 0$ such that

$$|h(t_1) - h(t_2)| \leq c_0 |t_1 - t_2|^\beta.$$

According to the definition given by Boufoussi et al. (2010), a Volterra-type multifractional Gaussian process, $\{B_h(t), t \geq 0\}$, with the Hurst functional parameter $h(t)$ is a centered Gaussian process defined by

$$B_h(t) := B_{h(t)}(t) = \int_0^t K_h(t, s)dW(s), \tag{1.5}$$

where W is a standard Brownian motion and the Volterra kernel is given by

$$K_h(t,s) := K_{h(t)}(t,s) = s^{\frac{1}{2}-h(t)} \int_s^t (y-s)^{h(t)-\frac{3}{2}} y^{h(t)-\frac{1}{2}} dy.$$

A multifractional Gaussian process reduces to a fractional Brownian motion (fBm) when the functional parameter $h(t)$ is a constant. In fact, the stochastic calculus with respect to fBm is now well established (we refer the reader to Coutin (2007) for a survey). But the stochastic calculus for multifractional Gaussian processes has not yet been developed explicitly. Also, the theory of stochastic delay differential equations with fractional noise has attracted a lot attentions of works from different approaches (see e.g. Ferrante and Rovira (2006), Neuenkirch et al. (2008), Ferrante and Rovira (2010), Caraballo et al. (2011), León and Tindel (2012), Boufoussi and Hajji (2012)) and the generalized delay logistic equation (1.4) with fractional Brownian noise can be considered as a special form of equations studied by Neuenkirch et al. (2008). But, to the best of our knowledge, studies of the delay differential equations with multifractional noise are scarce.

Recently, Boufoussi et al. (2010) have used the stochastic calculus developed by Alòs et al. (2001) to get a definition for stochastic integral with respect to $B_h(t)$. Our work follows the work of Boufoussi et al. and is organized as follows: In Section 2, we give a modification of the stochastic integral with respect to $B_h(t)$ defined in Boufoussi et al. (2010). Section 3 contains the main result of this paper: we introduce an approximate method for proving the existence of a positive solution of (1.4).

2. Preliminaries

The aim of this section is to introduce a stochastic integral with respect to Volterra-type multifractional Gaussian processes, B_h, with the Hurst parameter $h(t) \in [a,b] \subset (1/2, 1)$. For this purpose, we additionally assume that $h(t)$ is a continuously differentiable function on $[0, +\infty)$.

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ generated by Brownian motion W. For any $p \geq 1$, we denote by

$\mathbb{D}_W^{1,p}$ the space of Malliavin differentiable random variables with the norm

$$\|F\|_{1,p}^p := E|F|^p + \int_0^T E|D_r^W F|^p dr,$$

where $D_r^W F$ is the Malliavin derivative of F.

As shown by Boufoussi et al. (2010), the Gaussian property of $B_h(t)$ allows us to apply the stochastic calculus developed by Alòs et al. (2001) to get a stochastic integral with respect to $B_h(t)$.

Denote by $\mathbb{D}_W^{1,2}(\mathcal{H}_{Kr})$ the space of stochastic processes satisfying the following two conditions:

$$E\int_0^T \Big(\int_s^T |u(t)\partial_1 K_h(t,s)|dt \Big)^2 ds < \infty, \tag{C1}$$

and

$$E\int_0^T \int_0^T \Big(\int_s^T |D_r^W u(t)\partial_1 K_h(t,s)|dt \Big)^2 dsdr < \infty. \tag{C2}$$

where

$$\partial_1 K_h(t,s) = (t-s)^{h(t)-\frac{3}{2}} \left(\frac{t}{s}\right)^{h(t)-\frac{1}{2}} + h'(t)\int_s^t [\ln(y-s) + \ln\frac{y}{s}](y-s)^{h(t)-\frac{3}{2}} \left(\frac{y}{s}\right)^{h(t)-\frac{1}{2}} dy.$$

For $u = \{u(t), t \in [0,T]\}$ is a stochastic process in $\mathbb{D}_W^{1,2}(\mathcal{H}_{Kr})$, one can define the divergence integral of u with respect to B_h by (see formula (21) in Alòs et al. (2001))

$$\int_0^t u(s)\delta B_h(s) = \int_0^t \Big(\int_r^t u(s)\partial_1 K_h(s,r)ds \Big) \delta W(r),$$

where δW_r is Skorohod differential. Moreover, $r \mapsto \int_r^t u(s)\partial_1 K_h(s,r)ds$ is Stratonovich integrable with respect to W. By taking into account the relation between the Skorohod integral and the Stratonovich integral, we use in this paper the following definition for the multifractional stochastic integral.

Definition 2.1. Let $u \in \mathbb{D}_W^{1,2}(\mathcal{H}_{Kr})$. The stochastic integral of u with respect to B_h is defined by

$$\int_0^t u(s)dB_h(s) = \int_0^t u(s)\delta B_h(s) + \int_0^t \int_s^t D_s^W u(r)\, \partial_1 K_h(r,s) dr ds,$$

Proposition 2.1. *Let $u = \{u(t), 0 \le t \le T\}$ be a stochastic process bounded in the norm $\|.\|_{1,2}$ of the space $\mathbb{D}_W^{1,2}$, i.e.*

$$\sup_{0 \le t \le T} \left(E|u(t)|^2 + \int_0^T E|D_r^W u(t)|^2 dr \right) < \infty.$$

Then $u \in \mathbb{D}_W^{1,2}(\mathcal{H}_{Kr})$.

Proof. By using the Hölder inequality it is easy to check the conditions (C1) and (C2). □

3. The main results

In this section, we prove the existence of a semi-analytical solution to the equation (1.4). To obtain the solution, like in the classical case, we shall use the method of step by step, i.e. first prove the result for the interval $[0, r]$, then we use this solution process as the initial condition to solve the equation within the interval $[r, 2r]$, and so on. Thus the key problem is studying

$$dN(t) = \left(\lambda N(t) - C(t)[N(t)]^2 \right) dt + \sigma N(t) dB_h(t) \ , \ t \in [0,T], \tag{3.1}$$

where the initial condition $N(0) = N_0$ and $C(t)$ is an adapted stochastic process.

In the Brownian case, where $B_h(t)$ is replaced by a standard Brownian motion $W(t)$, the existence and uniqueness of the solution are well-known and the explicit solution can be found by using the Itô differential formula (for instance, see Kloeden and Platen (1992), page 125). In our context, $B_h(t)$ with $h(t) > \frac{1}{2}$ is not a semimartingale and (3.1) is an anticipated stochastic differential equation (it contains a Skorohod differential). Hence, the traditional methods cannot be applied. Naturally, we need to find a new method to studying (3.1). Let us state the following auxiliary result.

Theorem 3.1. **I.** *$\{B_h(t),\ 0 \le t \le T\}$ can be approximated in $L^p(\Omega), p > 1$ by semimartingale $B_{h,\varepsilon}(t)$ which is defined as follows for every $\varepsilon > 0$*

$$B_{h,\varepsilon}(t) := E[B_h(t+\varepsilon)|\mathcal{F}_t] = \int_0^t K_h(t+\varepsilon, s)dW(s), \tag{3.2}$$

where $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$ is the natural filtration associated to W.

II. *Let $u = \{u_t, 0 \le t \le T\}$ be an $\mathbb{F}$-adapted process bounded in the norm $\|.\|_{1,2}$ of the space $\mathbb{D}^{1,2}_W$. Then*

$$\int_0^t u(s)dB_h(s) = \lim_{\varepsilon \to 0^+} \int_0^t u(s)dB_{h,\varepsilon}(s) \ \ in \ \ L^2(\Omega),$$

uniformly in $t \in [0,T]$.

Proof. **I.** We first show that $B_{h,\varepsilon}(t)$ is a semimartingale. Indeed, by using the semimartingale decomposition for Volterra stochastic integrals we have

$$\int_0^t g(t,s)dW_s = \int_0^t g(s,s)dW_s + \int_0^t \Big(\int_0^s \frac{\partial}{\partial s} g(s,u)dW_u \Big) ds,$$

provided that integrals in the right hand side exist. And then by choosing $g(t,s) = K_h(t+\varepsilon, s)$ we get

$$B_{h,\varepsilon}(t) = \int_0^t K_h(s+\varepsilon, s)dW_s + \int_0^t \varphi_\varepsilon(s)ds \ , t \in [0,T], \tag{3.3}$$

where $\varphi_\varepsilon(s) = \int_0^s \partial_1 K_h(s+\varepsilon, u)dW_u$ and

$$\partial_1 K_h(s+\varepsilon, u) = (s+\varepsilon-u)^{h(s+\varepsilon)-\frac{3}{2}} \Big(\frac{s+\varepsilon}{u}\Big)^{h(s+\varepsilon)-\frac{1}{2}}$$
$$+ h'(s+\varepsilon) \int_u^{s+\varepsilon} [\ln(y-u) + \ln\frac{y}{u}](y-u)^{h(s+\varepsilon)-\frac{3}{2}} \Big(\frac{y}{u}\Big)^{h(s+\varepsilon)-\frac{1}{2}} dy.$$

(We note that $B_h(t)$ is not a semimartingale because $\varphi_0(s)$ does not exist.)

By the Itô isometry formula we have

$$E|B_{h,\varepsilon}(t)-B_h(t)|^2=\int_0^t[K_h(t+\varepsilon,s)]^2ds-2\int_0^t K_h(t+\varepsilon,s)K_h(t,s)ds+\int_0^t[K_h(t,s)]^2ds$$

$$\leq\int_0^{t+\varepsilon}[K_h(t+\varepsilon,s)]^2ds-2\int_0^{t\wedge t+\varepsilon}K_h(t+\varepsilon,s)K_h(t,s)ds+\int_0^t[K_h(t,s)]^2ds=E|B_h(t+\varepsilon)-B_h(t)|^2. \tag{3.4}$$

We recall from Corollary 4 in Boufoussi et al. (2010) that

$$E|B_h(t_1)-B_h(t_2)|^2\leq C_T|t_1-t_2|^{2\min(\beta,a)}\ \forall\ t_1,t_2\in[0,T], \tag{3.5}$$

where $a=\min\limits_{t\in[0,T]}h_t$ and β is Hölder exponent of h.

By the Gaussian properties of $B_{h,\varepsilon}(t)$ and $B_h(t)$ combined with (3.4) and (3.5), we obtain that for each $p>1$, there exists $C_{T,p}$ such that

$$E|B_{h,\varepsilon}(t)-B_h(t)|^p\leq C_{T,p}\varepsilon^{p\min(\beta,a)},$$

which implies that $B_{h,\varepsilon}(t)$ converges to $B_h(t)$ in $L^p(\Omega), p>1$ when $\varepsilon\to 0$, uniformly in $t\in[0,T]$.

II. Note that $\int\limits_0^t u(s)\,dB_{h,\varepsilon}(s)$ is well defined because u is a square integrable and $\mathbb{F}$-adapted process. From the decomposition (3.3) and by the integration by parts formula for the Skorohod integral we have

$$\int_0^t u(s)\,dB_{h,\varepsilon}(s)=\int_0^t u(s)K_h(s+\varepsilon,s)dW_s+\int_0^t u(s)\int_0^s\partial_1K_h(s+\varepsilon,r)dW_rds$$

$$=\int_0^t u(s)K_h(s+\varepsilon,s)dW_s+\int_0^t\int_0^s u(s)\partial_1K_h(s+\varepsilon,r)\delta W_rds+\int_0^t\int_r^t D_ru(s)\partial_1K_h(s+\varepsilon,r)dsdr$$

$$=\int_0^t u(s)K_h(s+\varepsilon,s)dW_s+\int_0^t\int_r^t u(s)\partial_1K_h(s+\varepsilon,r)ds\delta W_r+\int_0^t\int_r^t D_ru(s)\partial_1K_h(s+\varepsilon,r)dsdr.$$

As a consequence,

$$E\Big|\int_0^t u(s)\,dB_{h,\varepsilon}(s)-\int_0^t u(s)\,dB_h(s)\Big|^2 \le 3E\Big|\int_0^t u(s)K_h(s+\varepsilon,s)dW_s\Big|^2$$

$$+3E\Big|\int_0^t\int_r^t u(s)(\partial_1K_h(s+\varepsilon,r)-\partial_1K_h(s,r))ds\delta W_r\Big|^2$$

$$+3E\Big|\int_0^t\int_r^t D_r^W u(s)(\partial_1K_h(s+\varepsilon,r)-\partial_1K_h(s,r))dsdr\Big|^2 := 3(A_1+A_2+A_3).$$

It is obvious that $A_1\to 0$ as $\varepsilon\to 0$, uniformly in $t\in[0,T]$. By Meyer's inequality (see, Nualart (2006)) we have

$$A_2\le\int_0^t\Big\|\int_r^t u(s)(\partial_1K_h(s+\varepsilon,r)-\partial_1K_h(s,r))ds\Big\|_{1,2}^2dr,$$

which implies that $A_2\to 0$ uniformly in $t\in[0,T]$ as $\varepsilon\to 0$ because the process $u(s)$ is bounded in the norm $\|.\|_{1,2}$ and $h(t)$ is a differentiable function with bounded derivative on the interval $[0,T]$. Similarly, we also have $A_3\to 0$ because

$$A_3\le\Big\|\int_r^t u(s)(\partial_1K_h(s+\varepsilon,r)-\partial_1K_h(s,r))ds\Big\|_{1,2}^2.$$

The Theorem is proved. □

In order to be able to apply Theorem 3.1 and to use the method of induction in Theorem 3.3, let us introduce a new space: For fixed $p\ge 2$, denote by $\mathfrak{D}^{1,p}$ the space of stochastic processes such that $u\in\mathbb{D}_W^{1,p}$ and

$$\sup_{t\in[0,T]}E|u(t)|^p+\sup_{r,t\in[0,T]}E|D_r^Wu(t)|^p\le L_p,$$

where L_p is a finite positive constant. By Proposition 2.1 we have $\mathfrak{D}^{1,p}\subset\mathbb{D}_W^{1,2}(\mathcal{H}_K)$.

Theorem 3.2. *Suppose that $C(t)$ is an $\mathbb{F}$-adapted stochastic process such that $C(t) \geq 0$ a.s. for any $t \in [0,T]$ and $C \in \mathfrak{D}^{1,2+\delta}$ for some $\delta > 0$. Then the equation (3.1) has a positive solution which is given by*

$$N^*(t) = \frac{N_0 \exp(\lambda t + \sigma B_h(t))}{1 + N_0 \int_0^t C(s) \exp(\lambda s + \sigma B_h(s)) ds}. \tag{3.6}$$

This solution is adapted to the filtration $\mathbb{F}$ and belongs to $\mathfrak{D}^{1,2+\frac{\delta}{4}}$.

Proof. The proof is broken up into three steps.

Step 1. Consider the "approximation" equation corresponding to (3.1) with the same initial condition $N_\varepsilon(0) = N_0$:

$$dN_\varepsilon(t) = \big(\lambda N_\varepsilon(t) - C(t)[N_\varepsilon(t)]^2\big)dt + \sigma N_\varepsilon(t) dB_{h,\varepsilon}(t). \tag{3.7}$$

From the decomposition (3.3), the above equation can be rewritten as follows

$$dN_\varepsilon(t) = \bigg((\lambda + \sigma\varphi_\varepsilon(t))N_\varepsilon(t) - C(t)[N_\varepsilon(t)]^2\bigg)dt + \sigma K_h(t+\varepsilon,t)N_\varepsilon(t)dW_t. \tag{3.8}$$

To find the solution of (3.7) we put $Y_\varepsilon(t) := \exp\big(-\lambda t + \frac{1}{2}\sigma^2 \int_0^t [K_h(s+\varepsilon,s)]^2 ds - \sigma B_{h,\varepsilon}(t)\big)$, which solves the following Itô SDE

$$dY_\varepsilon(t) = \big(-\lambda + \sigma^2[K_h(t+\varepsilon,t)]^2 - \sigma\varphi_\varepsilon(t)\big)Y_\varepsilon(t)dt - \sigma K_h(t+\varepsilon,t)Y_\varepsilon(t)dW(t).$$

Since $N_\varepsilon(t)$ and $Y_\varepsilon(t)$ are semimartingales, we can apply the integration by parts formula to $Z_\varepsilon(t) := N_\varepsilon(t)Y_\varepsilon(t)$ and get

$$dZ_\varepsilon(t) = -\frac{C(t)}{Y_\varepsilon(t)}[Z_\varepsilon(t)]^2 dt\ ,\ Z_0^\varepsilon = N_0. \tag{3.9}$$

Obviously, the unique solution of (3.9) is given by $Z_\varepsilon(t) = \bigg(\frac{1}{N_0} + \int_0^t \frac{C(s)}{Y_\varepsilon(s)} ds\bigg)^{-1}$.

Consequently, the solution $N_\varepsilon(t) = \frac{Z_\varepsilon(t)}{Y_\varepsilon(t)}$ of (3.7) is given by

$$N_\varepsilon(t) = \frac{N_0 e^{\lambda t - \frac{1}{2}\sigma^2 \int_0^t [K_h(s+\varepsilon,s)]^2 ds + \sigma B_{h,\varepsilon}(t)}}{1 + N_0 \int_0^t C(s) e^{\lambda s - \frac{1}{2}\sigma^2 \int_0^s [K_h(u+\varepsilon,u)]^2 du + \sigma B_{h,\varepsilon}(s)} ds} := \frac{\bar{Y}_\varepsilon(t)}{\bar{Z}_\varepsilon(t)}.$$

Step 2. Check $N_\varepsilon, N^* \in \mathfrak{D}^{1,2}$. We observe that $N^*(t) = N_0(t)$, and hence, it is enough to show that $N_\varepsilon \in \mathfrak{D}^{1,2}\ \forall\ \varepsilon \geq 0$. Denote by $M_i(.,.), i = 1,2,...,$ the finite positive constants not depending on ε. By the chain rule of the Malliavin derivative we have for any $r \leq t$,

$$D_r^W[\bar{Y}_\varepsilon(t)] = \sigma\bar{Y}_\varepsilon(t)D_r^W[B_{h,\varepsilon}(t)] = \sigma\bar{Y}_\varepsilon(t)K_h(t+\varepsilon, r),$$

$$D_r^W[\bar{Z}_\varepsilon(t)] = \int_r^t \left(D_r^W[C(s)]\bar{Y}_\varepsilon(s) + C(s)D_r^W[\bar{Y}_\varepsilon(s)]\right)ds,$$

$$D_r^W[N_\varepsilon(t)] = \frac{\bar{Z}_\varepsilon(t)D_r^W[\bar{Y}_\varepsilon(t)] - \bar{Y}_\varepsilon(t)D_r^W[\bar{Z}_\varepsilon(t)]}{[\bar{Z}_\varepsilon(t)]^2}.$$

Note that $K_h(t,s) \leq \frac{(t-s)^{h(t)-\frac{1}{2}}}{h(t)-\frac{1}{2}} \leq \frac{T^{b-\frac{1}{2}}}{a-\frac{1}{2}}\ \forall\ 0 \leq s < t \leq T$. Then by the Gaussian property of $B_{h,\varepsilon}(t)$ we have the following two estimates for any $p > 1:$

$$E|\bar{Y}_\varepsilon(t)|^p = N_0^p e^{p(\lambda t - \frac{1}{2}\sigma^2\int_0^t K_h^2(s+\varepsilon,s)ds) + \frac{1}{2}p^2\sigma^2 E|B_{h,\varepsilon}(t)|^2} < M_1(p,T),$$

$$E|D_r^W[\bar{Y}_\varepsilon(t)]|^p = \sigma^p|K_h(t+\varepsilon,r)|^p E|\bar{Y}_\varepsilon(t)|^p < M_2(p,T).$$

By the Hölder inequality

$$\begin{aligned} E|\bar{Z}_\varepsilon(t)|^{2+\frac{\delta}{2}} &\leq 2 + 2\int_0^t E|C(s)\bar{Y}_\varepsilon(s)|^{2+\frac{\delta}{2}}ds \\ &\leq 2 + 2\int_0^t (E|C(s)|^{2+\delta})^{\frac{2+\frac{\delta}{2}}{2+\delta}}(E|\bar{Y}_\varepsilon(s)|^{\frac{(4+\delta)(2+\delta)}{\delta}})^{\frac{\delta}{4+2\delta}}ds < M_3(\delta,T). \end{aligned}$$

In a similar way we have $E|D_r^W[\bar{Z}_\varepsilon(t)]|^{2+\frac{\delta}{2}} < M_4(\delta,T)$ and

$$E|D_r^W[N_\varepsilon(t)]|^{2+\frac{\delta}{4}} \leq E|\bar{Z}_\varepsilon(t)D_r^W[\bar{Y}_\varepsilon(t)] - \bar{Y}_\varepsilon(t)D_r^W[\bar{Z}_\varepsilon(t)]|^{2+\frac{\delta}{4}} < M_5(\delta,T). \tag{3.10}$$

Since $N_0C(t) \geq 0$ *a.s.*, this implies that

$$E|N_\varepsilon(t)|^{2+\frac{\delta}{4}} \leq E|\bar{Y}_\varepsilon(t)|^{2+\frac{\delta}{4}} < M_1(\delta,T). \tag{3.11}$$

The inequalities (3.10) and (3.11) show that $N_\varepsilon \in \mathfrak{D}^{1,2+\frac{\delta}{4}} \subset \mathfrak{D}^{1,2}$.

Step 3. Firstly, it is easy to check that $N_\varepsilon(t) \to N^*(t)$ uniformly in $t \in [0,T]$ in the norm $\|.\|_{1,2}$ since $K_h(t,t) = 0$ and $B_{h,\varepsilon}(t) \to B_h(t)$ for any $t \in [0,T]$. On the other hand, $N^*(t)$ is $\mathbb{F}$-adapted. It follows from Theorem 3.1 and triangular inequality that

$$\int_0^t N_\varepsilon(s)dB_{h,\varepsilon}(s) \to \int_0^t N^*(s)dB_h(s) \text{ in } L^2(\Omega).$$

As a consequence, we can take the limit of the integral form of (3.7) in $L^2(\Omega)$ as $\varepsilon \to 0$ to get

$$N^*(t) = N_0 + \int_0^t \big(\lambda N^*(s) - C(s)[N^*(s)]^2\big)ds + \int_0^t \sigma N^*(s)dB_h(s)\ ,\ t \in [0,T],$$

which means that $N^*(t)$ is a solution of (3.1).

The Theorem is thus proved. $\square$

Theorem 3.3. *Consider the generalized delay logistic equations with multifractional noise (1.4). Suppose that $g(t,x)$ is a continuously differentiable function in x such that*

$$|g(t,x) - g(t,y)| < L|x-y|\ \forall\, t \in [0,T],\ \forall\, x,y \in \mathbb{R}, \tag{3.12}$$

$$|g(t,x)| < L(1+|x|)\ \forall\, t \in [0,T],\ \forall\, x \in \mathbb{R}. \tag{3.13}$$

In addition, $g(t,x) \geq 0$ for all $t \in [0,T]$ and $x \in \mathbb{R}$. Then the equation (1.4) has a positive solution on $[0,T]$.

Proof. Because the delay time, r, is discrete we can prove our theorem by the method of induction. For simplicity let us assume that $T = Nr$, where N is a positive integer number. Our induction hypothesis, for $n < N$, is the following:

(H_n) The equation

$$N(t) = \phi(0) + \int_0^t \big(\lambda N(s) - g(s, N(s-r))[N(s)]^2\big)ds + \int_0^t \sigma N(s)dB_h(s)\ ,\ t \in [0,nr],$$

with $X_t = 0\,, t > nr$, has a positive solution in $\mathfrak{D}^{1,2+\frac{1}{2^{2n}}}$.

Check (H_1). Let $t \in [0, r]$. Then $N(t-r) = \phi(t-r)$ and (1.4) becomes

$$N(t) = \phi(0) + \int_0^t (\lambda N(s) - C_1(s)[N(s)]^2)ds + \int_0^t \sigma N(s) dB_h(s). \tag{3.14}$$

where $C_1(s) = g(s, \phi(s-r)) \geq 0$. It is obvious that $C_1 \in \mathfrak{D}^{1,2+1}$ since ϕ is a deterministic function. From Theorem 3.2 we conclude that (H_1) is true.

Induction. Assume that (H_i) is true for $i \leq n$, with $n < N$. We wish to prove that (H_{n+1}) is also true. Consider the stochastic process defined as

$$V(t) = \begin{cases} \phi(t-r) & \text{if } t \leq r, \\ N(t-r) & \text{if } r < t \leq (n+1)r, \\ 0 & \text{if } t > (n+1)r, \end{cases}$$

where N is the solution obtained in (H_n). We have $V(t) \in \mathfrak{D}^{1,2+\frac{1}{2^{2n}}}$.

Put $C_n(s) = g(s, V_s) \geq 0$. Once again, we need to check $C_n \in \mathfrak{D}^{1,2+\frac{1}{2^{2n}}}$. Since $g(t,x)$ is a continuously differentiable function in x with bounded derivative, we have

$$D_r^W[C_n(t)] = \frac{\partial}{\partial x} g(t, V(t)) D_r^W[V(t)].$$

From (3.12) and (3.13) we have

$$|D_r^W[C_n(t)]| < L|D_r^W[V(t)]| \ ; \ |C_n(t)| < L(1 + |V(t)|),$$

which mean that $C_n \in \mathfrak{D}^{1,2+\frac{1}{2^{2n}}}$. So (H_{n+1}) is true.

The proof of Theorem is complete. □

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References

Alòs, E., Mazet, O., Nualart, D., 2001. Stochastic calculus with respect to Gaussian processes. Ann. Probab. 29 (2), 766-801.

Boufoussi, B., Dozzi, M., Marty, R., 2010. Local time and Tanaka formula for a Volterra-type multifractional Gaussian process. Bernoulli 16 (4), 1294-1311.

Boufoussi, B., Hajji, S., 2012. Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space. Statist. Probab. Lett. 82 (8),1549-1558.

Caraballo, T., Garrido-Atienza, M.J., Taniguchi, T., 2011. The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion. Nonlinear Anal. 74 (11), 3671-3684.

Coutin, L., 2007. An introduction to stochastic calculus with respect to fractional Brownian motion. In Séminaire de Probabilités XL, page 3-65. Springer-Verlag Berlin Heidelberg.

Ferrante, M., Rovira, C., 2006. Stochastic delay differential equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Bernoulli 12 (1), 85-100.

Ferrante, M., Rovira, C., 2010. Convergence of delay differential equations driven by fractional Brownian motion. J. Evol. Equ. 10 (4), 761-783.

Kloeden, P.E., Platen, E., 1992. Numerical solution of stochastic differential equations. Springer Verlag.

León, J., Tindel, S., 2012. Malliavin calculus for fractional delay equations. J. Theor. Probab. 25 (3), 854-889.

Neuenkirch, A., Nourdin, I., Tindel, S., 2008. Delay equations driven by rough paths. Electron. J. Probab. 13, 2031-2068.

Nualart, D., 2006. The Malliavin calculus and related topics. 2nd edition, Springer.

Yukalov, V.I., Yukalova, E.P., Sornette, D., 2009. Punctuated evolution due to delayed carrying capacity. Physica D 238, 1752-1767.

Verhulst, P.F., 1838. Notice sur la loi que la population suit dans son accroissement, Correspondence Math. Phys. 10, 113-121.