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Stochastic Volterra integro-differential equations driven by fractional Brownian motion in a Hilbert space

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In this article, we consider a class of stochastic Volterra integro-differential equations with infinite delay and impulsive effects, driven by fractional Brownian motion with the Hurst index $H > 1/2$ in a Hilbert space. The cases of Lipschitz and bounded impulses are studied separately. The existence and uniqueness of mild solutions are proved by using different fixed-point theorems. An example is given to illustrate the theory.

Keywords: fractional Brownian motion; Volterra equations; mild solutions; impulses; infinite delays

2000 AMS Classification number: 60H15; 60G22; 45D05

1. Introduction

The fractional Brownian motion (fBm) and its basic properties have been studied by Mandelbrot and Van Ness [18]. In the last decades, a lot of works have been carried out to develop stochastic calculus with respect to fBm. The rigorous definitions of stochastic integrals with respect to fBm and the theory of stochastic differential equations driven by fBm as well as its applications have been studied intensively. We refer the reader to two monographs [3,19] and the references therein for a more complete presentation of this subject.

It is known that the impulsive effects exist widely in many evolution processes in which states are changed abruptly at certain moments of time, involving such fields as telecommunications, neural networks, mechanics, electronics, and finance and economics (see e.g. [16]). Hence, it is quite natural to take into account the effect of impulses in the investigation of stochastic differential equations driven by fBm. However, to the best of our knowledge, no work has been reported in the present literature regarding the theory of stochastic differential equations driven by fBm with impulsive effects. The aim of this article is to study one such equation. Our work is inspired by the work of Caraballo et al. [6] where the following stochastic differential equation driven by fBm with finite delays has been studied:

$$\begin{cases} \mathrm{d}x(t) = \left[Ax(t) + f(t, x_t)\right]\mathrm{d}t + g(t)\,\mathrm{d}W^H(t), \quad t \in [0, T], \\ x(t) = \phi(t), \quad t \in [-\tau, 0](0 \le \tau < \infty). \end{cases}$$

In this article, we investigate the existence and uniqueness of mild solutions to semi-linear stochastic Volterra equations with infinite delays and impulses of the following form in a

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Hilbert space

$$\begin{cases} \mathrm{d}x(t) = \left[Ax(t) + F\left(t, x_t, \int_0^t k(t,s)x(s)\,\mathrm{d}s\right)\right]\mathrm{d}t + G(t)\,\mathrm{d}W^H(t), & t \in [0,T],\ t \neq t_k, \\ \Delta x(t_k) := x\left(t_k^+\right) - x\left(t_k^-\right) = I_k\left(x\left(t_k^-\right)\right), & k = 1,2,\ldots,m, \\ x(t) = \phi(t), & t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of an analytic semi-group of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X with norm $\|\cdot\|$, W^H is a fBm with $H > 1/2$ on a real and separable Hilbert space Y. The history $x_t : (-\infty, 0] \to X, x_t(\theta) = x(t+\theta), \theta \leq 0$ belongs to an abstract phase space $\mathcal{B}_h$. The Volterra kernel $k(t,s)$ is non-negative continuous function on $t \in [0, t_1]$, The functions $F : [0,T] \times \mathcal{B}_h \times X \to X$, $G : [0,T] \to \mathcal{L}_2^0(Y,X)$, and $I_k : X \to X$ are defined later. Furthermore, the impulsive moments satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $x(t_k^+), x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$. The spaces $\mathcal{B}_h$ and $\mathcal{L}_2^0(Y,X)$ are defined in Section 2.

It is known that fBm is a generalization of Brownian motion, it reduces to Brownian motion when $H = 1/2$. In fact, the existence and uniqueness of mild solutions to stochastic Volterra equations with delay and impulsive effects, driven by a Brownian process in Hilbert spaces are now well established (see e.g. [2,14,15,24,25] and the references therein), but the equations driven by fBm have not yet been fully developed. The Equation (1.1) belongs to the class of stochastic delay differential equations driven by fBm. This class is so new that only few works have appeared till date. The finite dimensional equations was first investigated by Ferrante and Rovira [10] and then by Neuenkirch et al. [20], Boufoussi and Hajji [4], Dung [8], León and Tindel [17], and some other authors. The case of the equations in a Hilbert space has been considered by Caraballo et al. [6] and by Boufoussi and Hajji [5]. The finite dimensional stochastic Volterra equations with delay have been recently studied by Dung [9]. We would like to emphasize that in most of these works, the delays are finite. Thus, the appearance of infinite delay and Volterra term in (1.1) as well as the study of the problem in a Hilbert space make our article more interesting even in the case without impulses.

This article is organized as follows. In Section 2, we recall the definition of the fractional Wiener integral with respect to an infinite dimensional fBm and the definition of mild solutions. Section 3 is devoted to study the existence and uniqueness of mild solutions when the impulses are Lipschitz. The case of bounded impulses is studied in Section 4. Conclusion and an example are provided in Section 5.

2. Preliminaries

In this section, we first recall the definition of Wiener integrals with respect to an infinite dimensional fBm with Hurst index $H > 1/2$. We also refer the reader to [7] for a detailed presentation of this integral and for a short review of the development of stochastic differential equations driven by fBm without impulses in a Hilbert space.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $T > 0$ be an arbitrary fixed horizon. A one-dimensional fBm with Hurst parameter $H \in (0,1)$ is a centred Gaussian process $\beta^H = \{\beta^H(t), 0 \leq t \leq T\}$ with the covariance function $R_H(t,s) = E[\beta^H(t)\beta^H(s)]$

$$R_H(t,s) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

It is known that $\beta^H(t)$ with $H > 1/2$ admits the following Volterra representation:

$$\beta^H(t) = \int_o^t K(t,s)\,\mathrm{d}\beta(s), \tag{2.1}$$

where β is a standard Brownian motion and the Volterra kernel $K(t,s)$ is given by

$$K(t,s) = c_H \int_s^t (u-s)^{H-(3/2)} \left(\frac{u}{s}\right)^{H-(1/2)} \mathrm{d}u, \qquad t \geq s.$$

For the deterministic function $\varphi \in L^2([0,T])$, it is known from [3,21] that the fractional Wiener integral of φ with respect to β^H can be defined by

$$\int_0^T \varphi(s)\,\mathrm{d}\beta^H(s) = \int_0^T K_H^* \varphi(s)\,\mathrm{d}\beta(s),$$

where $K_H^*\varphi(s) = \int_s^T \varphi(r)(\partial K/\partial r)(r,s)\,\mathrm{d}r$.

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operators from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y, and $\mathcal{L}(Y,X)$. Let $\{\mathrm{e}_n, n = 1, 2, \ldots\}$, be a complete orthonormal basis in Y and $Q \in \mathcal{L}(Y,X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\mathrm{tr}Q = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n, n = 1, 2, \ldots$ are non-negative real numbers. We define the infinite dimensional fBm on Y with covariance Q as

$$W^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where $\beta_n^H(t)$ are real, independent fBms. This process is a Y-valued Gaussian; it starts from 0 and has zero mean and covariance:

$$E\langle W^H(t), x\rangle\langle W^H(s), y\rangle = R(t,s)\langle Q(x), y\rangle \quad \text{for all } x, y \in Y \quad \text{and} \quad t, s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ of all Q-Hilbert–Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert–Schmidt operator if

$$\|\psi\|_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

The fractional Wiener integral of the function $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ with respect to Q-fBm is defined by

$$\int_0^t \psi(s)\,\mathrm{d}W^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\psi(s) e_n \mathrm{d}\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} K_H^*(\psi e_n)(s)\,\mathrm{d}\beta_n(s), \tag{2.2}$$

where β_n is the standard Brownian motion used to present β_n^H as in (2.1).

Noting that unlike the classical Wiener integral, the Itô isometry formula is not available for the fractional Wiener integral. However, we have the following fundamental inequality which was proved in [6].

Lemma 2.1. If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s < \infty$ then the sum in (2.2) is well defined as a X-valued random variable and we have

$$E\left\|\int_0^t \psi(s)\,\mathrm{d}W^H(s)\right\|^2 \leq 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s.$$

It is known that the study of the theory of differential equation with infinite delays depends on a choice of the abstract phase space (see [12]). Let us present an abstract space phase. Assume that $h : (-\infty, 0] \to (0,\infty)$ be a continuous function with $\int_{-\infty}^0 h(t)\,\mathrm{d}t < \infty$. We define the abstract phase space $\mathcal{B}_h$ by

$$\mathcal{B}_h = \Big\{\phi : (-\infty,0] \to X : \text{for any } \ a > 0, (E\|\phi\|^2)^{1/2} \ \text{ is bounded and measurable} \\ \text{function on } \ [-a,0] \ \text{ with } \ \phi(0) = 0 \ \text{ and } \ \int_{-\infty}^0 h(t) \sup_{t\leq\theta\leq 0}(E\|\phi\|^2)^{1/2}\,\mathrm{d}t < \infty\Big\}.$$

If we equip the space $\mathcal{B}_h$ with the norm

$$\|\phi\|_{\mathcal{B}_h} := \int_{-\infty}^0 h(t) \sup_{t\leq\theta\leq 0}(E\|\phi(\theta)\|^2)^{1/2}\mathrm{d}t,$$

then $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space [13].

We now consider the space $\mathcal{B}_{DI}$ (D and I stand for delay and impulse, respectively) defined by

$$\begin{aligned}\mathcal{B}_{DI} =&\{x : (-\infty,T] \to X : x|I_k \in C(I_k,X) \ \text{ and } \ x\big(t_k^+\big), x\big(t_k^-\big) \ \text{ exist with} \\ &x\big(t_k^-\big) = x(t_k), x_0 = \phi \in \mathcal{B}_h, k = 1,2,\ldots,m\},\end{aligned} \tag{2.3}$$

where $x|_{I_k}$ is the restriction of x to the interval $I_k = (t_k, t_{k+1}], k = 0,1,\ldots,m$. The function $\|\cdot\|_{\mathcal{B}_{DI}}$ is a semi-norm in $\mathcal{B}_{DI}$, it is defined by

$$\|x\|_{\mathcal{B}_{DI}} = \|x_0\|_{\mathcal{B}_h} + \sup_{0\leq t\leq T}(E\|x(t)\|^2)^{1/2}.$$

The following lemma is a common property of phase spaces. It can be easily found by a simple computation.

Lemma 2.2. Suppose that $x \in \mathcal{B}_{DI}$, then $x_t \in \mathcal{B}_h$ for all $t \in [0,T]$ and

$$\|x_t\|_{\mathcal{B}_h} \leq \|x_0\|_{\mathcal{B}_h} + a \sup_{0\leq s\leq t}(E\|x(s)\|^2)^{1/2},$$

where $a = \int_{-\infty}^0 h(t)\,\mathrm{d}t$.

We end this section by giving the definition of mild solutions for the Equation (1.1) which is similar to the deterministic situation. For simplicity, we can assume that $x(0) = \phi(0) = 0$.

Definition 2.1. A X-valued stochastic process $\{x(t), t \in (-\infty, T]\}$ is called a mild solution of the Equation (1.1) if $x_0 = \phi \in \mathcal{B}_h$ on $(-\infty, 0]$ with $\phi(0) = 0$ and the following conditions hold

(i) for each $t \in [0, T]$, x_t is a $\mathcal{B}_h$-valued function and the restriction of $x(\cdot)$ to the interval $(t_k, t_{k+1}], k = 1, 2, \ldots, m$ is continuous,
(ii) for each $t \in [0, T]$, we have a.s.

$$x(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ \int_0^t S(t-s)F(s, x_s, \int_0^s k(s,u)x(u)\mathrm{d}u)\mathrm{d}s + \int_0^t S(t-s)G(s)\mathrm{d}W^H(s), \quad t \in [0, t_1], \\ S(t-t_k)\big(x\big(t_k^-\big)\big) + I_k\big(x\big(t_k^-\big)\big) + \int_{t_k}^t S(t-s)F(s, x_s, \int_0^s k(s,u)x(u)\mathrm{d}u)\mathrm{d}s \\ \quad + \int_{t_k}^t S(t-s)G(s)\mathrm{d}W^H(s), \ t \in (t_k, t_{k+1}], \ k = 1, 2, \ldots, m, \end{cases} \tag{2.4}$$

(iii) for each k, the limits $x(t_k^+), x(t_k^-)$ exist with $x(t_k^-) = x(t_k)$ and $\Delta x(t_k) = I_k(x(t_k^-))$.

3. Equations with Lipschitz impulses

In this section, we investigate the existence and uniqueness of mild solutions when the impulsive functions are Lipschitz continuous. In order to prove the required results, we assume the following conditions:

(H_1) A is the infinitesimal generator of an analytic semi-group, $(S(t))_{t \geq 0}$, of bounded linear operators on X. Moreover, $S(t)$ satisfies the condition that there exists a positive constant M such that for $t \in [0, T]$

$$\|S(t)\| \leq M.$$

(H_2) There exist $L_1, L_2 > 0$ such that

$$E\|F(t, \psi, x) - F(t, \varphi, y)\|^2 \leq L_1\|\psi - \varphi\|^2_{\mathcal{B}_h} + L_2 E\|x - y\|^2$$

for all $t \in [0, T]$, $\psi, \varphi \in \mathcal{B}_h$ and $x, y \in L^2(\Omega, X)$.
(H_3) For each $k = 1, 2, \ldots, m$, there exist a constant $\rho_k > 0$ such that

$$\|I_k(x) - I_k(y)\|^2 \leq \rho_k\|x - y\|^2$$

for all $x, y \in X$.
(H_4) The function $G : [0, T] \to \mathcal{L}_2^0(Y, X)$ satisfies

$$\int_0^T \|G(s)\|^2_{\mathcal{L}_2^0} \mathrm{d}s < \infty.$$

Theorem 3.1. Assume that the conditions $(H_1) - (H_4)$ hold. Then, the Equation (1.1) has a unique mild solution, provided that

$$\max_{k=1,2,\ldots,m} \big(3M^2\big(1 + \rho_k + T^2(L_1 a^2 + L_2 K^*)\big)\big) < 1, \tag{3.1}$$

where

$$K^* = \left(\sup_{0 \le t \le T} \int_0^t k(t,s)\,\mathrm{d}s \right)^2$$

Proof. Our proof is based on the Banach contraction principle. To do this, we define the operator Φ on $\mathcal{B}_{DI}$ by

$$(\Phi x)(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ \int_0^t S(t-s)F(s, x_s, \int_0^s k(s,u)x(u)\,\mathrm{d}u)\,\mathrm{d}s + \int_0^t S(t-s)G(s)\,\mathrm{d}W^H(s), \quad t \in [0, t_1], \\ S(t-t_k)\big(x\big(t_k^-\big) + I_k\big(x\big(t_k^-\big)\big)\big) + \int_{t_k}^t S(t-s)F(s, x_s, \int_0^s k(s,u)x(u)\,\mathrm{d}u)\,\mathrm{d}s \\ \quad + \int_{t_k}^t S(t-s)G(s)\,\mathrm{d}W^H(s), \quad t \in (t_k, t_{k+1}], \quad k = 1, 2, \ldots, m. \end{cases} \tag{3.2}$$

From the conditions $(H_1) - (H_4)$, it can be seen that Φ maps $\mathcal{B}_{DI}$ into itself. Let $y : (-\infty, T] \to X$ be the function defined by

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ 0, & t \in [0, T], \end{cases}$$

then $y_0 = \phi$. For each $z : [0, T] \to X$ with $z(0) = 0$, we define the function $\bar{z}$ by

$$\bar{z}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ z(t), & t \in [0, T]. \end{cases}$$

If $x(\cdot)$ satisfies (2.4) then we can decompose $x(t)$ into $x(t) = y(t) + \bar{z}(t), t \in (-\infty, T]$. This implies that $x_t = y_t + \bar{z}_t$ and the function $z(\cdot)$ satisfies

$$z(t) = \begin{cases} \int_0^t S(t-s)F(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u)\,\mathrm{d}s \\ + \int_0^t S(t-s)G(s)\,\mathrm{d}W^H(s), \quad t \in [0, t_1], \\ S(t-t_k)\big(z\big(t_k^-\big) + I_k\big(z\big(t_k^-\big)\big)\big) \\ + \int_{t_k}^t S(t-s)F(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u)\,\mathrm{d}s \\ + \int_{t_k}^t S(t-s)G(s)\,\mathrm{d}W^H(s), \quad t \in (t_k, t_{k+1}], \quad k = 1, 2, \ldots, m. \end{cases} \tag{3.3}$$

Set $\mathcal{B}_{DI}^0 = \{z \in \mathcal{B}_{DI} : z_0 = 0\}$ and let $\|\cdot\|_{\mathcal{B}_{DI}^0}$ be the norm defined by

$$\|z\|_{\mathcal{B}_{DI}^0} = \|z_0\|_{\mathcal{B}_h} + \sup_{0 \le t \le T} (E\|z(t)\|^2)^{1/2} = \sup_{0 \le t \le T} (E\|z(t)\|^2)^{1/2}.$$

Thus, $(\mathcal{B}^0_{DI}, \|\cdot\|_{\mathcal{B}^0_{DI}})$ is a Banach space. Define $\Psi : \mathcal{B}^0_{DI} \to \mathcal{B}^0_{DI}$ by

$$(\Psi z)(t) = \begin{cases} \int_0^t S(t-s)F(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u)+\bar{z}(u))\,\mathrm{d}u)\,\mathrm{d}s \\ \quad +\int_0^t S(t-s)G(s)\,\mathrm{d}W^H(s), \quad t \in [0, t_1], \\ S(t-t_k)\big(z\big(t_k^-\big) + I_k\big(z\big(t_k^-\big)\big)\big) \\ +\int_{t_k}^t S(t-s)F(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u)+\bar{z}(u))\,\mathrm{d}u)\,\mathrm{d}s \\ +\int_{t_k}^t S(t-s)G(s)\,\mathrm{d}W^H(s), \quad t \in (t_k, t_{k+1}], \quad k = 1, 2, \ldots, m. \end{cases} \tag{3.4}$$

It is clear that the operator Φ has a unique fixed-point if and only if Ψ has a unique fixed point. Thus, it is sufficient to show that Ψ is a contraction map. Let $z, z* \in \mathcal{B}^0_{DI}$, then for all $t \in [0, t_1]$ we have

$$\begin{aligned} E\|(\Psi z)(t) - (\Psi z^*)(t)\|^2 \le & E\bigg\|\int_0^t S(t-s)\bigg(F(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u)+\bar{z}(u))\,\mathrm{d}u\bigg) \\ & -F\bigg(s, y_s + \bar{z}^*_s, \int_0^s k(s,u)(y(u)+\bar{z}^*(u))\,\mathrm{d}u\bigg)\mathrm{d}s\bigg\|^2 \\ & \le M^2 T\int_0^t \bigg(L_1\|\bar{z}_s - \bar{z}^*_s\|^2_{\mathcal{B}_h} \\ & +L_2 E\|\int_0^s k(s,u)[y(u)+\bar{z}(u)-y(u)-\bar{z}^*(u)]\mathrm{d}u\|^2\bigg)\mathrm{d}s. \end{aligned}$$

By using Lemma 2.2, we get

$$\begin{aligned} E\|(\Psi z)(t) - (\Psi z^*)(t)\|^2 & \le M^2 T\int_0^t \bigg(L_1 a^2 \sup_{0\le u\le s} E\|z(u) - z^*(u)\|^2 \\ & +L_2 K^* \sup_{0\le u\le s} E\|z(u) - z^*(u)\|^2\bigg)\mathrm{d}s \\ & \le M^2 T^2 (L_1 a^2 + L_2 K^*)\|z - z^*\|^2_{\mathcal{B}^0_{DI}}, \quad \forall t \in [0, t_1]. \end{aligned} \tag{3.5}$$

For $t \in (t_1, t_2]$, in the similar way to the above estimate, we have

$$\begin{aligned} & E\bigg\|\int_{t_1}^t S(t-s)\bigg(F(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u)+\bar{z}(u))\,\mathrm{d}u\bigg) \\ & -F\bigg(s, y_s + \bar{z}^*_s, \int_0^s k(s,u)(y(u)+\bar{z}^*(u))\,\mathrm{d}u\bigg)\bigg\|^2 \mathrm{d}s \le M^2 T^2\big(L_1 a^2 + L_2 K^*\big)\|z - z^*\|^2_{\mathcal{B}^0_{DI}}. \end{aligned}$$

Hence,

$$\begin{aligned}E\|(\Psi z)(t)-(\Psi z^*)(t)\|^2 &\leq 3E\left\|S(t-t_1)\left(z\left(t_1^-\right)-z^*\left(t_1^-\right)\right)\right\|^2\\&\quad+3E\left\|S(t-t_1)\left(I_1\left(z\left(t_1^-\right)\right)-I_1\left(z^*\left(t_1^-\right)\right)\right)\right\|^2\\&\quad+3E\left\|\int_{t_1}^t S(t-s)\left(F\left(s,y_s+\bar{z}_s,\int_0^s k(s,u)(y(u)+\bar{z}(u))\,\mathrm{d}u\right)\right.\right.\\&\quad\left.\left.-F\left(s,y_s+\bar{z}_s^*,\int_0^s k(s,u)(y(u)+\bar{z}^*(u)\right)\mathrm{d}u)\right)\right\|^2\mathrm{d}s\\&\leq 3M^2(1+\rho_1)E\|z(t_1^-)-z^*(t_1^-)\|^2\\&\quad+3M^2T^2(L_1a+L_2K^*)\|z-z^*\|^2_{\mathcal{B}^0_{DI}}.\end{aligned}$$

As a consequence,

$$E\|(\Psi z)(t)-(\Psi z^*)(t)\|^2\leq 3M^2\left(1+\rho_1+T^2(L_1a^2+L_2K^*)\right)\|z-z^*\|^2_{\mathcal{B}^0_{DI}},\quad \forall t\in(t_1,t_2].$$

Similarly, when $t\in(t_k,t_{k+1}]$, $k=2,3,\ldots,m$, we also have

$$E\|(\Psi z)(t)-(\Psi z^*)(t)\|^2\leq 3M^2\left(1+\rho_k+T^2(L_1a^2+L_2K^*)\right)\|z-z^*\|^2_{\mathcal{B}^0_{DI}},\quad \forall t\in(t_k,t_{k+1}].$$

Thus, for all $t\in[0,T]$

$$E\|(\Psi z)(t)-(\Psi z^*)(t)\|^2\leq \max_{k=1,2,\ldots,m}\left(3M^2\left(1+\rho_k+T^2\left(L_1a^2+L_2K^*\right)\right)\right)\|z-z^*\|^2_{\mathcal{B}^0_{DI}}.$$

This, together with the condition (3.1), implies that Ψ is a contraction map and, therefore, it has a unique fixed point $z\in\mathcal{B}^0_{DI}$.

The proof is complete. □

We end this section by showing the existence and uniqueness of a mild solution for a stochastic evolution equation without impulses which has been discussed by Caraballo et al. [6] when the delay is finite.

Corollary 3.1. Assume that the conditions (H_1) and (H_4) hold and that there exists $L_1>0$ such that

$$E\|F(t,\psi)-F(t,\varphi)\|^2\leq L_1\|\psi-\varphi\|^2_{\mathcal{B}_h}$$

for all $t\in[0,T]$, $\psi,\varphi\in\mathcal{B}_h$. Then, the stochastic evolution equation with infinite delay

$$\begin{cases}\mathrm{d}x(t)=\left[Ax(t)+F(t,x_t)\right]\mathrm{d}t+G(t)\,\mathrm{d}W^H(t),\quad t\in[0,T],\\ x(t)=\phi(t),\quad t\in(-\infty,0],\end{cases}\tag{3.6}$$

admits a unique mild solution for any initial data $\phi\in\mathcal{B}_h$.

4. Equations with bounded impulses

The aim of this section is to prove the existence and uniqueness of mild solutions when the impulsive functions are bounded. Let us introduce two new assumptions.

(H_5) $F : [0,T]\times\mathcal{B}_h\times X \to X$ is continuous, and there exist two continuous functions $\mu_1, \mu_2, \mu_3 : [0,T] \to (0,\infty)$ such that

$$E\|F(t,\varphi,x)\|^2 \leq \mu_1(t)\|\varphi\|^2_{\mathcal{B}_h} + \mu_2(t)E\|x\|^2 + \mu_3(t)$$

for all $t \in [0,T]$, $\varphi \in \mathcal{B}_h$ and $x \in L^2(\Omega, X)$.
(H_6) $I_k : X \to X$, $k = 1,2,\ldots,m$ are continuous and there exist finite positive constants d_k such that $\|I_k(x)\| \leq \mathrm{d}_k$ for all $x \in X$.

Because of the lack of the Lipschitz property of the impulsive functions, it seems to be impossible to use the contraction mapping principle in proving the existence and uniqueness of the solution. The main result of this section is based on the following fixed-point theorem (see, for instance, [23]).

Lemma 4.1 (Schaefers fixed point theorem). Let $(D, \|\cdot\|)$be a normed space, and let the operator $A : D \to D$ be a continuous map which is compact on each bounded subset of D. Define

$$\mathcal{S}(A) = \{x \in D : x = \lambda Ax, \lambda \in (0,1)\}.$$

Then, either

(i) the set $\mathcal{S}(A)$ is unbounded, or
(ii) the operator A has a fixed point in D.

Theorem 4.1. Suppose that (H_1) and $(H_4) - (H_6)$ hold. Then, the Equation (1.1) has at least a mild solution. Furthermore, if (H_2) holds, then the solution is unique.

Proof. Before giving a proof of the results, let us show a useful estimate which is based on the condition (H_5)

$$\begin{aligned}
&E\left\|F\left(s, y_s+\bar{z}_s, \int_0^s k(s,u)(y(u)+\bar{z}(u))\,\mathrm{d}u\right)\right\|^2 \\
&\quad\leq \mu_1(s)\|y_s+\bar{z}_s\|^2_{\mathcal{B}_h} + \mu_2(s)E\left\|\int_0^s k(s,u)(y(u)+\bar{z}(u))\,\mathrm{d}u\right\|^2 + \mu_3(s) \\
&\quad\leq 2\mu_1(s)\left(\|\phi\|^2_{\mathcal{B}_h} + a^2\sup_{0\leq u\leq s} E\|z(u)\|^2\right) + \mu_2(s)K^*\sup_{0\leq u\leq s} E\|z(u)\|^2 + \mu_3(s) \\
&\quad\leq 2\mu_1^*\left(\|\phi\|^2_{\mathcal{B}_h} + a^2\sup_{0\leq u\leq s} E\|z(u)\|^2\right) + \mu_2^*K^*\sup_{0\leq u\leq s} E\|z(u)\|^2 + \mu_3^*,
\end{aligned} \tag{4.1}$$

where $\mu_i^* = \sup_{0\leq s\leq T} \mu_i(s)$, $i = 1,2,3$.

Existence. We define the operator $\Psi : \mathcal{B}_{DI}^0 \to \mathcal{B}_{DI}^0$ as in Theorem 4.2. In order to be able to use Lemma 4.1, we separate the proof into four steps:

Step 1. We first show that the subset

$$\mathcal{S}(\Psi) := \left\{z \in \mathcal{B}_{DI}^0 : z = \lambda\Psi(z), \lambda \in (0,1)\right\}$$

is bounded. Let $z \in \mathcal{S}(\Psi)$, then $z = \lambda\Psi(z)$ for some $\lambda \in (0,1)$. Then, for each $t \in [0, t_1]$, we have

$$z(t) = \lambda\int_0^t S(t-s)F\left(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u\right)\mathrm{d}s + \lambda\int_0^t S(t-s)G(s)\,\mathrm{d}W^H(s).$$

This, together with the condition (H_1) and Lemma 2.1, implies that

$$\begin{aligned} E\|z(t)\|^2 &< E\left\|\int_0^t S(t-s)F\left(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u\right)\mathrm{d}s \right.\\ &\qquad \left. + \int_0^t S(t-s)G(s)\,\mathrm{d}W^H(s)\right\|^2 \\ &\leq 2M^2\left[t\int_0^t E\left\|F\left(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u\right)\right\|^2 \mathrm{d}s \right.\\ &\qquad \left. + 2Ht^{2H-1}\int_0^t \|G(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s\right]. \end{aligned}$$

For $t \in (t_1, t_2]$, we have

$$\begin{aligned} z(t) &= \lambda S(t-t_1)\left(z\left(t_1^-\right) + I_1\left(z\left(t_1^-\right)\right)\right) \\ &\quad + \lambda\int_{t_1}^t S(t-s)F\left(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u\right)\mathrm{d}s + \lambda\int_{t_1}^t S(t-s)G(s)\,\mathrm{d}W^H(s) \\ &= \lambda S(t-t_1)I_1\left(z\left(t_1^-\right)\right) + \lambda\int_0^t S(t-s)F\left(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u\right)\mathrm{d}s \\ &\quad + \lambda\int_0^t S(t-s)G(s)\,\mathrm{d}W^H(s). \end{aligned}$$

Hence,

$$\begin{aligned} E\|z(t)\|^2 \leq 3M^2&\left[\mathrm{d}_1^2 + t\int_0^t E\left\|F\left(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u\right)\right\|^2 \mathrm{d}s \right.\\ &\quad \left. + 2Ht^{2H-1}\int_0^t \|G(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s\right]. \end{aligned}$$

Similarly for $t \in (t_k, t_{k+1}]$, $k = 2, 3, ..., m$, we can obtain that

$$E\|z(t)\|^2 \leq 3M^2\left[\left(\sum_{k=1}^{m} \mathrm{d}_k\right)^2 + t\int_0^t E\left\|F\left(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u\right)\right\|^2 \mathrm{d}s\right.$$
$$\left. + 2Ht^{2H-1}\int_0^t \|G(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s\right], \quad \forall t \in [0,T].$$

We now use the estimate (4.1) to get

$$\sup_{0\leq u\leq t} E\|z(u)\|^2 \leq 3M^2\left[T\int_0^t \left(2\mu_1^*\left(\|\phi\|^2_{\mathcal{B}_h} + a^2 \sup_{0\leq u\leq s} E\|z(u)\|^2\right) + \mu_2^* K^* \sup_{0\leq u\leq s} E\|z(u)\|^2\right)\mathrm{d}s\right.$$
$$\left. + \mu_3^* T^2 + 2HT^{2H-1}\int_0^T \|G(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s + \left(\sum_{k=1}^{m} \mathrm{d}_k\right)^2\right], \quad \forall t \in [0,T].$$

An application of Gronwall's lemma to the above inequality yields

$$\sup_{0\leq u\leq t} E\|z(u)\|^2 \leq 3M^2\left[2T^2\mu_1^*\|\phi\|^2_{\mathcal{B}_h} + \mu_3^* T^2 + 2HT^{2H-1}\int_0^T \|G(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s\right.$$
$$\left. + \left(\sum_{k=1}^{m} \mathrm{d}_k\right)^2\right] \mathrm{e}^{3M^2T(2\mu_1^* a^2 + \mu_2^* K^*)t}, \quad \forall t \in [0,T].$$

So, the set $\mathcal{S}(\Psi)$ is bounded.

Step 2. Let $B_q = \{z \in \mathcal{B}^0_{DI} : \|z\|_{\mathcal{B}^0_{DI}} \leq q\}$. We will show Ψ maps bounded sets B_q into equicontinuous sets. Let $u, v \in [0, t_1]$, without loss of generality, we can assume that $u \leq v$. Then,

$$(\Psi z)(v) - (\Psi z)(u) = \int_0^u (S(v-s) - S(u-s))F\left(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u\right)\mathrm{d}s$$
$$+ \int_u^v S(v-s)F(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u)\,\mathrm{d}s$$
$$+ \int_0^u (S(v-s) - S(u-s))G(s)\,\mathrm{d}W^H(s) + \int_u^v S(v-s)G(s)\,\mathrm{d}W^H(s) :$$
$$= Q_1 + Q_2 + Q_3 + Q_4.$$

We, therefore, have

$$E\|(\Psi z)(v) - (\Psi z)(u)\|^2 \leq 4(E\|Q_1\|^2 + E\|Q_2\|^2 + E\|Q_3\|^2 + E\|Q_4\|^2), \tag{4.2}$$

where

$$
\begin{aligned}
E\|Q_1\|^2 &= E\left\|\int_0^u (S(v-s)-S(u-s))F\left(s, y_s+\bar{z}_s, \int_0^s k(s,u)(y(u)+\bar{z}(u))\,\mathrm{d}u\right)\mathrm{d}s\right\|^2 \\
&\le \int_0^u \|S(v-s)-S(u-s)\|^2\,\mathrm{d}s\int_0^u E\left\|F\left(s, y_s+\bar{z}_s, \int_0^s k(s,u)(y(u)+\bar{z}(u))\,\mathrm{d}u\right)\right\|^2\mathrm{d}s \\
&\le \int_0^u \|S(v-s)-S(u-s)\|^2\,\mathrm{d}s\int_0^u \left(2\mu_1^*\left(\|\phi\|^2_{\mathcal{B}_h}+a^2q^2\right)+\mu_2^*K^*q^2+\mu_3^*\right)\mathrm{d}s \\
&\le \left(2\mu_1^*\left(\|\phi\|^2_{\mathcal{B}_h}+a^2q^2\right)+\mu_2^*K^*q^2+\mu_3^*\right)T\int_0^u \|S(v-s)-S(u-s)\|^2\,\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
E\|Q_2\|^2 &= E\left\|\int_u^v S(v-s)F\left(s, y_s+\bar{z}_s, \int_0^s k(s,u)(y(u)+\bar{z}(u))\,\mathrm{d}u\right)\mathrm{d}s\right\|^2 \\
&\le \left(2\mu_1^*\left(\|\phi\|^2_{\mathcal{B}_h}+a^2q^2\right)+\mu_2^*K^*q^2+\mu_3^*\right)(v-u)\int_u^v \|S(v-s)\|^2\,\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
E\|Q_3\|^2 &= E\left\|\int_0^u (S(v-s)-S(u-s))G(s)\,\mathrm{d}W^H(s)\right\|^2 \\
&\le 2Hu^{2H-1}\int_0^u \|(S(v-s)-S(u-s))G(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
E\|Q_4\|^2 &= E\|\int_u^v S(v-s)G(s)\,\mathrm{d}W^H(s)\|^2 \\
&\le 2H(v-u)^{2H-1}\int_u^v \|S(v-s)G(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s \\
&\le 2M^2H(v-u)^{2H-1}\int_u^v \|G(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s.
\end{aligned}
$$

Obviously, $E\|Q_2\|^2 \to 0$ and $E\|Q_4\|^2 \to 0$ as $u \to v$. Since $S(t)$ is strongly continuous, this implies that $E\|Q_1\|^2 \to 0$ as $u \to v$. Moreover, $\|(S(v-s)-S(u-s))G(s)\|^2_{\mathcal{L}_2^0} \le 4M^2\|G(s)\|^2_{\mathcal{L}_2^0} \in L^1([0,T])$, we also have $E\|Q_3\|^2 \to 0$ by the dominated convergence theorem.

Similarly for $u, v \in (t_k, t_{k+1}], k = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
E\|(\Psi z)(v)-(\Psi z)(u)\|^2 &\le 6E\left\|[S(v-t_k)-S(u-t_k)]z\left(t_k^-\right)\right\|^2 \\
&\quad + 6E\left\|[S(v-t_k)-S(u-t_k)]I_k\left(z\left(t_k^-\right)\right)\right\|^2 \\
&\quad + 6\left(E\|Q_1\|^2+E\|Q_2\|^2+E\|Q_3\|^2+E\|Q_4\|^2\right) \\
&\le 6q^2E\|S(v-t_k)-S(u-t_k)\|^2 \\
&\quad + 6\mathrm{d}_k^2E\|S(v-t_k)-S(u-t_k)\|^2 \\
&\quad + 6\left(E\|Q_1\|^2+E\|Q_2\|^2+E\|Q_3\|^2+E\|Q_4\|^2\right).
\end{aligned}
\tag{4.3}
$$

The right hand sides of (4.2) and (4.3) do not depend on $x \in B_q$ and converge to 0 when $u \to v$. This proves that Ψ maps bounded sets into equicontinuous family of functions.

Step 3. We now prove that Ψ is an compact operator. Using the same arguments as in Step 1, we can get

$$E\|(\Psi z)(t)\|^2 \leq 3M^2\Bigg[T\int_0^t\left(2\mu_1^*\left(\|\phi\|^2_{\mathcal{B}_h}+a^2\sup_{0\leq u\leq s}E\|z(u)\|^2\right)+\mu_2^*K^*\sup_{0\leq u\leq s}E\|z(u)\|^2\right)\mathrm{d}s$$
$$+\mu_3^*T^2+2HT^{2H-1}\int_0^T\|G(s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s+\left(\sum_{k=1}^m \mathrm{d}_k\right)^2\Bigg],\quad \forall t\in[0,T],$$

which points out that if $z\in B_q$, then $\Psi z\in B_{q'}$ for some q'. Thus, Ψ maps bounded sets into bounded sets in $\mathcal{B}^0_{DI}$. This fact, combined with Step 2, means that the set $\{\Psi(z)\times(t): z\in B_q\}$ is relatively compact in B^0_{DI}. Hence, Ψ is a compact operator by the Arzelà–Ascoli theorem.

Step 4. We finally show that Ψ is continuous. Let $z\in\mathcal{B}^0_{DI}$ and $\{z^n\}_{n\geq1}$ be a sequence in $\mathcal{B}^0_{DI}$ such that $\|z^n-z\|\to0$ as $n\to\infty$. Obviously, there exists an integer number q such that $z_n, z\in B_q$ for all $n\geq1$. Denote

$$F^n(s)=F\left(s, y_s+\bar{z}^n_s, \int_0^s k(s,u)(y(u)+\bar{z}^n(u))\,\mathrm{d}u\right)$$
$$-F\left(s, y_s+\bar{z}_s, \int_0^s k(s,u)(y(u)+\bar{z}(u))\,\mathrm{d}u\right).$$

Since F is continuous on $[0,T]\times\mathcal{B}_h\times X\to X$, this implies that $F^n(s)\to0$ as $n\to\infty$. Moreover, by the estimate (4.1), we have

$$E\|F^n(s)\|^2\leq 8\mu_1^*\left(\|\phi\|^2_{\mathcal{B}_h}+a^2q^2\right)+4\mu_2^*K^*q^2+4\mu_3^*.$$

Now, for every $t\in[0,t_1]$, we have

$$E\|(\Psi z^n)(t)-(\Psi z)(t)\|^2\leq E\|\int_0^t S(t-s)F^n(s)\,\mathrm{d}s\|^2\leq M^2T\int_0^t E\|F^n(s)\|^2\,\mathrm{d}s,$$

which means that $E\|(\Psi z^n)(t)-(\Psi z)(t)\|^2\to0$ by the dominated convergence theorem.

Since $I_k, k=1,2,\ldots,m$ are continuous functions, we also have the following convergence for $t\in(t_k,t_{k+1}], k=1,2,\ldots,m$

$$E\|(\Psi z^n)(t)-(\Psi z)(t)\|^2\leq 3M^2\Big(E\big\|z^n(t_k^-)-z(t_k^-)\big\|^2+E\big\|I_k(z^n(t_k^-))-I_k(z(t_k^-))\big\|^2$$
$$+M^2T\int_{t_k}^t E\|F^n(s)\|^2\,\mathrm{d}s\Big)\to0,\quad n\to\infty.$$

Thus, Ψ is continuous and by Lemma 4.1 the operator Ψ has a fixed point. Hence, we can conclude that the Equation (1.1) has at least a mild solution.

Uniqueness. From the above proofs, we see that the conditions $(H_1), (H_4), (H_5)$ and (H_6) ensure the existence of the mild solution of (1.1). We will use the condition (H_2) to show that the solution is unique.

Let z, z^* be two mild solutions of (3.3) with the same initial condition. On the interval $[0, t_1]$, we have from (3.5) that

$$\begin{aligned} E\|z(t) - z^*(t)\|^2 &= E\|(\Psi z)(t) - (\Psi z^*)(t)\|^2 \\ &\leq M^2 T\left(L_1 a^2 + L_2 K^*\right)\int_0^t \sup_{0\leq u\leq s} E\|z(u) - z^*(u)\|^2 \,\mathrm{d}s, \end{aligned}$$

which implies that

$$\sup_{0\leq u\leq t} E\|z(u) - z^*(u)\|^2 \leq M^2 T(L_1 a^2 + L_2 K^*)\int_0^t \sup_{0\leq u\leq s} E\|z(u) - z^*(u)\|^2 \,\mathrm{d}s. \tag{4.4}$$

An application of Gronwall's lemma to (4.4) yields $E\|x_1(t) - x_2(t)\|^2 \leq 0$. This proves that $z(t) = z^*(t)$a.s. for all $t \in [0, t_1]$.

On the interval $(t_1, t_2]$, we have

$$\begin{aligned} z(t) = &\, S(t-t_1)\big(y(t_1^-) + \bar{z}(t_1^-) + I_k\big(y(t_1^-) + \bar{z}(t_1^-)\big)\big) \\ &+ \int_{t_1}^t S(t-s)F(s, y_s + \bar{z}_s, \int_0^s k(s,u)(y(u) + \bar{z}(u))\,\mathrm{d}u)\,\mathrm{d}s + \int_{t_1}^t S(t-s)G(s)\,\mathrm{d}W^H(s), \end{aligned}$$

and

$$\begin{aligned} z^*(t) = &\, S(t-t_1)\Big(y(t_1^-) + \bar{z}^*(t_1^-) + I_k\Big(y(t_1^-) + \bar{z}^*(t_1^-)\Big)\Big) \\ &+ \int_{t_1}^t S(t-s)F\left(s, y_s + \bar{z}_s^*, \int_0^s k(s,u)(y(u) + \bar{z}^*(u))\,\mathrm{d}u\right)\mathrm{d}s + \int_{t_1}^t S(t-s)G(s)\,\mathrm{d}W^H(s). \end{aligned}$$

From the fact $z(t_1^-) = z^*(t_1^-)$, we see that z, z^* are also two mild solutions with the same initial condition on the interval $(t_1, t_2]$. Thus, by Gronwall's lemma, we also have $z(t) = z^*(t)$a.s. for all $t \in (t_1, t_2]$. Similarly, we can conclude that $z(t) = z^*(t)$a.s. for all $t \in [0, T]$.

Since $x(t) = y(t) + \bar{z}(t), t \in (-\infty, T]$, the proof of Theorem 4.1 is complete. □

Corollary 4.1. Assume that the conditions (H_1) and (H_4) hold and that the function $F : [0, T] \times \mathcal{B}_h \to X$ is continuous, and there exists continuous functions $\mu_1, \mu_2 : [0, T] \to (0, \infty)$ such that

$$E\|F(t, \varphi)\|^2 \leq \mu_1(t)\|\varphi\|^2_{\mathcal{B}_h} + \mu_2(t)$$

for all $t \in [0, T]$, $\varphi \in \mathcal{B}_h$. Then, the stochastic evolution equation with infinite delay (3.6) has at least a mild solution for any initial data $\phi \in \mathcal{B}_h$.

5. Conclusion and examples

In this article, we proved the existence and uniqueness of the mild solution to a class of stochastic Volterra equations with infinite delay and impulsive effects, driven by an fBm with $H > 1/2$. Theorem 3.1 showed that the mild solution uniquely exists if the drift function F and impulsive functions are Lipschitz continuous. In addition, when the impulsive functions are bounded we only need one more condition on linear growth of the drift function to ensure the unique existence of the mild solution (Theorem 4.1). Thus, under some suitable assumptions, if the drift function F is Lipschitz and has linear growth then the appearance of impulses does not affect the existence and uniqueness of the mild solution.

Our obtained results extend the results of Caraballo et al. [6] to the case of infinite delay. In this sense, we partly enrich the knowledge of the theory of stochastic evolution equations driven by fBm. We would also like to remark that the results of this article are still true if we replace Q-fBm by a more general Q-Gaussian process as long as the stochastic integral of $G(t)$ with respect to this Gaussian process is well defined and has the finite second moments. For example, one of such Q-Gaussian processes can be constructed as follows: we consider one-dimensional Gaussian processes of the form

$$\theta_n(t) := \int_0^t K(t,s)\,\mathrm{d}\beta_n(s),$$

where $K(t,s)$ is a Volterra kernel satisfying the condition (K4) required by Alòs et al. [1]. Then, we can define a Q-Gaussian process by $\Theta(t) := \sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n\theta_n(t)$, and the integral of $G(t)$ with respect to $\Theta(t)$ by

$$\int_0^t G(s)\,\mathrm{d}\Theta(t) = \sum_{n=1}^{\infty}\int_0^t \sqrt{\lambda_n}G(s)e_n\,\mathrm{d}\theta_n(s),$$

where the stochastic integrals on the right hand side are given the formula (21) in [1].

We end this article with an example. It is known that the study of stochastic equations in a Hilbert space is important because of its close connection to the theory of finite-dimensional stochastic partial differential equations (see e.g. [11]). To illustrate the obtained theory, let us consider a stochastic partial differential equation with impulsive effects of the following form:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \int_{-\infty}^t H(t,x,s-t)Q(u(s,x))\,\mathrm{d}s + \int_0^t k(t,s)u(s,x)\,\mathrm{d}s \\ \qquad\qquad + G(t)\frac{\mathrm{d}W^H}{\mathrm{d}t}(t), \quad 0 \le t \le T, \ \ t \neq t_k, \ \ 0 \le x \le \pi; \\ \\ \Delta u(t_k,x) = \int_{-\infty}^{t_k} q_k(t_k - s)g(u(s,x))\,\mathrm{d}s, \quad k = 1,2,\ldots,m; \\ u(t,0) = u(t,\pi) = 0, \quad 0 \le t \le T; \\ u(t,x) = \phi(t,x), \quad -\infty < t \le 0, \ \ 0 \le x \le \pi; \end{cases} \tag{5.1}$$

where $W^H(t)$ is a cylindrical fBm and the function G satisfies the condition (H_4).

Let $X = L^2([0,\pi])$ with the norm $\|\cdot\|$ and inner product $\langle .,. \rangle$. Define $A : X \to X$ by $Az = z''$ with domain

$$\mathcal{D}(A) := \{z \in X : z, z' \text{are absolutely continuous } z'' \in X, z(0) = z(\pi) = 0\}.$$

Then,

$$Az = \sum_{n=1}^{\infty} n^2\langle z, z_n\rangle, \quad z \in \mathcal{D}(A),$$

where $z_n(t) = \sqrt{2/\pi}\sin(nt)$, $n = 1, 2, \ldots$ is the orthogonal set of eigenvectors in A. It is well known that A is the infinitesimal generator of an analytic semi-group $(S(t))_{t\geq 0}$ in X. Furthermore, we have (see [22])

$$S(t)z = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2 t}\langle z, z_n\rangle z_n, \quad \text{for all } \ z \in X \ \text{ and every } \ t > 0.$$

Since the analytic semi-group $S(t)$ is compact, there exists a constant M such that $\|S(t)\| \leq M$. In other words, the condition (H_1) holds.

We choose the phase function $h(s) = \mathrm{e}^s$, $s \leq 0$, then $a = \int_{-\infty}^{0} h(s)\,\mathrm{d}s = 1 < \infty$, and the abstract phase space $\mathcal{B}_h$ is Banach with the norm

$$\|\phi\|_{\mathcal{B}_h} := \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} (E\|\phi(\theta)\|^2)^{1/2}\,\mathrm{d}s.$$

For $(t, \phi) \in [0, T] \times \mathcal{B}_h$, where $\phi(\theta)(x) = \phi(\theta, x), (\theta, x) \in (-\infty, 0] \times [0, \pi]$, we put $u(t) \times (x) = u(t, x)$ and define the function $F : [0, T] \times \mathcal{B}_h \times X \to X$ for the infinite delay as follows:

$$F\left(t, \phi, \int_0^t k(t, s)u(s)\,\mathrm{d}s\right)(x) = \int_{-\infty}^{0} H(t, x, \theta)Q(\phi(\theta)(x))\,\mathrm{d}\theta + \int_0^t k(t, s)u(s, x)\,\mathrm{d}s.$$

Then, with these settings, Equation (5.1) can be written in the form of Equation (1.1)

We now assume that the functions $q_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$ are continuous and $\mathrm{d}_k := \int_{-\infty}^{0} h(s)q_k^2(s)\,\mathrm{d}s < \infty$. Then, the condition (H_6) is satisfied if $g(\cdot)$ is continuous and bounded and the condition (H_3) holds if $g(\cdot)$ is Lipschitz continuous. To verify the conditions (H_3) and (H_5), we suppose further that

(i) the function $H(t, x, \theta)$ is continuous in $[0, T] \times [0, \pi] \times (-\infty, 0]$ and satisfies

$$\int_0^{\pi} \left(\int_{-\infty}^{0} |H(t, x, \theta)|\mathrm{d}\theta\right)^2 \mathrm{d}x := p(t) < \infty.$$

(ii) the function $Q(\cdot)$ is continuous and $EQ^2(\phi(\theta)(x)) \leq \|\phi\|_{\mathcal{B}_h}^2$ for all $(\theta, x) \in (-\infty, 0] \times [0, \pi]$.

(iii) the function $Q(\cdot)$ is continuous and $E|Q(\phi(\theta)(x)) - Q(\varphi(\theta)(x))|^2 \leq \|\phi - \varphi\|_{\mathcal{B}_h}^2$ for all $(\theta, x) \in (-\infty, 0] \times [0, \pi]$.

We can see from (i) and (ii) that

$$\begin{aligned}\left\|F\left(t, \phi, \int_0^t k(t,s)u(s)\mathrm{d}s\right)\right\|^2 &= \int_0^{\pi}\left(\int_{-\infty}^{0} H(t,x,\theta)Q(\phi(\theta)(x))\mathrm{d}\theta + \int_0^t k(t,s)u(s,x)\mathrm{d}s\right)^2 \mathrm{d}x \\ &\leq 2\int_0^{\pi}\left(\int_{-\infty}^{0} |H(t,x,\theta)|\mathrm{d}\theta\right)\left(\int_{-\infty}^{0} |H(t,x,\theta)|Q^2(\phi(\theta)(x))\mathrm{d}\theta\right)\mathrm{d}x \\ &\quad + 2\int_0^{\pi}\left(\int_0^t k(t,s)u(s,x)\mathrm{d}s\right)^2 \mathrm{d}x.\end{aligned}$$

Hence,

$$E\left\|F\left(t,\phi,\int_0^t k(t,s)u(s)\,\mathrm{d}s\right)\right\|^2 \leq 2\int_0^{\pi}\left(\int_{-\infty}^0 |H(t,x,\theta)|\mathrm{d}\theta\right)^2 \mathrm{d}x\|\phi\|^2_{\mathcal{B}_h} + 2E\left\|\int_0^t k(t,s)u(s)\,\mathrm{d}s\right\|^2,$$

which implies that (H_5) is satisfied with $\mu_1(t) = 2p(t)$, $\mu_2(t) = 2$, $\mu_3(t) = 0$. Similarly, by (i) and (iii) the condition (H_2) is fulfilled with $L_1 = 2p^*$, where $p^* = \sup_{0\leq t\leq T} p(t)$ and $L_2 = 2$.

Since the conditions of either Theorem 3.1 or Theorem 4.1 are fulfilled, we can conclude that Equation (5.1) has a unique mild solution on $(-\infty, T] \times [0, \pi]$.

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