FRACTIONAL GEOMETRIC MEAN REVERSION PROCESSES

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ABSTRACT. We use a method developed in Carmona et al. (2003) [2] to study the fractional Geometric mean reversion processes. Our obtained results hold for any $H \in (\frac{1}{4}, 1)$.
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1. INTRODUCTION

It is known that, for commodities, interest rates and exchange rates, a financial mean-reversion model has more economic logic than the geometric Brownian model. The basic geometric mean-reversion model is of the following form

$$dX_t = k(\mu - \ln X_t)X_t dt + \sigma X_t dB_t, \tag{1.1}$$

where μ is the long-run equilibrium level (of a stock price X_t, say), k is the speed of reversion, and B_t is a standard Brownian motion.

However, many observations show that an asset price or an interest rate is not always a Markov process since it has long-range aftereffects. And in this context, it is suitable to express it as a dynamics driven by a fractional Brownian motion.

In this paper, we study a class of fractional geometric mean reversion processes expressed by a fractional stochastic differential equation (SDE) of the form

$$\begin{cases} dX_t = (\mu_t - k_t \ln X_t)X_t dt + \sigma_t X_t dW_t^H & , \ 0 \leq t \leq T, \\ X_0 = x > 0, \end{cases} \tag{1.2}$$

where W_t^H is a fractional Brownian motion of the Liouville form. This can be considered as a generalization of many important financial models such as that of Black-Scholes, and of (1.1) which was used by Tvedt [11] to model spot freight in shipping.

The fractional Brownian motion (fBm) of the Liouville form with Hurst index $H \in (0,1)$ is a centered Gaussian process defined by

$$W_t^H = \int_0^t K(t,s)dB_s \tag{1.3}$$

where B is a standard Brownian motion and the kernel $K(t,s) = (t-s)^{\alpha}$, $\alpha = H - \frac{1}{2}$.

In the case where $H = \frac{1}{2}$, W^H is a standard Brownian motion and for $H \neq \frac{1}{2}, W^H$ is neither a semimartingale nor a Markov process. Hence, the stochastic calculus developed by Itô cannot be applied.

In this paper we use an approximate approach introduced by Tran Hung Thao and Christine Thomas-Agnan [8] with the fundamental result saying that a fBm can be L^2-approximated by semimartingales. This approach was used by Thao to study the fractional Ornstein-Uhlenbeck process and fractional Black-Scholes model [8, 9, 10] and then by N. T. Dung [4] to solve a class of fractional SDE's with polynomial drift. In those papers, authors used the definition of the fractional stochastic integral as a limit in $L^2(\Omega)$ of stochastic integral with respect to semimartingale, if it exists and their results hold only when $H > \frac{1}{2}$.

In [2], Carmona, Coutin and Montseny have given an sufficient condition (see hypothesis **(H)** below) for existence of limit in $L^2(\Omega)$ and so the fractional stochastic integral can be explicitly represented via the Skorohod integral and the Malliavin derivative.

Our paper follows Carmona, Coutin and Montseny's work and is organized as follows: In Section 2, we restate some basic facts about a semimartingale approximation of fractional processes and the definition of fractional integral. Section 3 contains main result of this paper that the explicit solution of (1.2) is found. Section 4 contains some comments.

2. Preliminaries

Theorem 2.1. *The fractional Brownian motion* $\{W_t^H, 0 \le t \le T\}$ *can be approximated uniformly in* t *in* $L^2(\Omega)$ *by the processes*

$$W_t^{H,\varepsilon} = \int_0^t K(t+\varepsilon, s)dB_s\,, \quad \varepsilon > 0.$$

$W_t^{H,\varepsilon}$ *is* $\mathcal{F}_t$*-semimartingale with following decomposition*

$$W_t^{H,\varepsilon} = \int_0^t K(s+\varepsilon, s)dB_s + \int_0^t \varphi_s^\varepsilon ds = \varepsilon^\alpha B_t + \int_0^t \varphi_s^\varepsilon ds, \tag{2.1}$$

where $(\mathcal{F}_t, 0 \le t \le T)$ *is the natural filtration associated to* B *or* W^H *and*

$$\varphi_s^\varepsilon = \int_0^s \partial_1 K(s+\varepsilon, u)dB_u\,,\ \partial_1 K(t,s) = \alpha(t-s)^{\alpha-1}.$$

Proof. A detail proof of this theorem can be found in [10]. □

From now we denote by **(H)** the space of stochastic processes satisfying the following hypothesis:

Hypothesis (H): Assume that f is an adapted process belonging to the space $\mathbf{D}_B^{1,2}$ and that there exists β fulfilling $\beta + H > 1/2$ and $p > 1/H$ such that

(i) $\|f\|^2_{L^{1,2}_\beta} := \sup\limits_{0<s<u<T} \frac{E\left[(f_u-f_s)^2 + \int_0^T (D_r^B f_u - D_r^B f_s)^2 dr\right]}{|u-s|^{2\beta}}$ is finite,

(ii) $\sup\limits_{0<s<T} f_s$ belongs to $L^p(\Omega)$.

Remark 2.1. The space $\mathbf{D}_B^{1,2}$ is defined as follows:

For $h \in L^2([0,T], \mathbb{R})$, we denote by $B(h)$ the Wiener integral

$$B(h) = \int_0^T h(t)dB_t.$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega,\mathcal{F},P)$ consisting of those classes of random variables of the form

$$F = f(B(h_1),...,B(h_n)), \tag{2.2}$$

where $n\in\mathbb{N}, f\in C_b^\infty(\mathbb{R}^n, L^2([0,T],\mathbb{R})), h_1,...,h_n\in L^2([0,T],\mathbb{R})$. If F has the form (2.2), we define its derivative as the process $D^BF := \{D_t^BF, t\in[0,T]\}$ given by

$$D_t^BF = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(B(h_1),...,B(h_n))h_k(t).$$

We shall denote by $\mathbf{D}_B^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} := \big[E|F|^2\big]^{\frac{1}{2}} + E\Big[\int_0^T |D_u^BF|^2du\Big]^{\frac{1}{2}}.$$

It is well known from [2] that for an adapted process f belonging to the space $\mathbf{D}_B^{1,2}$ we have

$$\begin{aligned}\int_0^t f_s\,dW_s^{H,\varepsilon} &= \int_0^t f_sK(s+\varepsilon,s)\,dB_s + \int_0^t f_s\varphi_s^\varepsilon ds \\ &= \int_0^t f_sK(t+\varepsilon,s)\,dB_s + \int_0^t\int_s^t (f_u-f_s)\,\partial_1K(u+\varepsilon,s)du\delta B_s \\ &\qquad + \int_0^t\int_0^s D_u^Bf_s\,\partial_1K(s+\varepsilon,u)duds\,,\end{aligned} \tag{2.3}$$

where the second integral in the right-hand side is a Skorohod integral (we refer to [7] for more detail about the Skorohod integral). For $f\in$ **(H)**, $\int_0^t f_s\,dW_s^{H,\varepsilon}$ converges in $L^2(\Omega)$ as $\varepsilon\to 0$. Each term in the right-hand side of (2.3) converges to the same term where $\varepsilon=0$. Then, it is "natural" to define

Definition 2.1. Let $f \in$ **(H)**. The fractional stochastic integral of f with respect to W^H is defined by

$$\int_0^t f_s \, dW_s^H = \int_0^t f_s K(t,s)\, dB_s + \int_0^t \int_s^t (f_u - f_s)\, \partial_1 K(u,s) du \delta B_s + \int_0^t du \int_0^u D_s^B f_u \, \partial_1 K(u,s) ds \,. \tag{2.4}$$

3. The main result

Our main contribution here is to introduce an approximation equation for the fractional geometric mean reversion process X_t, to find its solution X_t^ε and to prove the uniqueness of this solution. Also the solution of the initial problem is shown to be exactly the L^2-limit of X_t^ε when $\varepsilon \to 0$.

Let us consider the semilinear differential equation in a complete probability space $(\Omega, \mathcal{F}, P)$

$$dX_t = (\mu_t - k_t \ln X_t) X_t dt + \sigma_t X_t dW_t^H \ , \ t \in [0,T], \tag{3.1}$$

where the coefficients μ_t, k_t, σ_t are the deterministic functions and the initial condition $X_0 = x$ is a positive constant. Since the Malliavin derivative $D_u^B f_s = 0$ for any deterministic function f_s we have the following definition

Definition 3.1. The solution of (3.1) is a stochastic process belonging to the space **(H)** and that has a form

$$X_t = X_0 + \int_0^t (\mu_s - k_s \ln X_s) X_s ds + \int_0^t \sigma_s X_s K(t,s)\, dB_s + \int_0^t \int_s^t (\sigma_u X_u - \sigma_s X_s)\, \partial_1 K(u,s) du \delta B_s + \int_0^t \int_0^s \sigma_s D_u^B X_s \, \partial_1 K(s,u) du ds \,. \tag{3.2}$$

Since the equation (3.2) contains the Skorohod integral and the Malliavin derivative, we cannot apply standard methods (for instance, Picard iteration procedure) to prove the existence and uniqueness of the solution. However, the fact that W_t^H can be approximated uniformly in $t \in [0,T]$ by semimartingales $W_t^{H,\varepsilon}$ leads us to consider the approximation equation corresponding to (3.2)

$$X_t^\varepsilon = X_0 + \int_0^t (\mu_s - k_s \ln X_s^\varepsilon) X_s^\varepsilon ds + \int_0^t \sigma_s X_s^\varepsilon K(t+\varepsilon, s)\, dB_s + \int_0^t \int_s^t (\sigma_u X_u^\varepsilon - \sigma_s X_s^\varepsilon)\, \partial_1 K(u+\varepsilon, s) du \delta B_s + \int_0^t \int_0^s \sigma_s D_u^B X_s^\varepsilon\, \partial_1 K(s+\varepsilon, u) du ds\,, \tag{3.3}$$

Noting that in **(H)**, the solution X_t^ε of (3.3) converges in $L^2(\Omega)$ to the solution X_t of (3.2). Thus, we can find the solution of (3.2) by solving the approximation equation (3.3) in **(H)** and then taking limit in $L^2(\Omega)$ as $\varepsilon \to 0$.

Now we can rewrite the approximation equation (3.3) as follows

$$dX_t^\varepsilon = (\mu_t - k_t \ln X_t^\varepsilon) X_t^\varepsilon dt + \sigma_t X_t^\varepsilon dW_t^{H,\varepsilon}$$

or

$$dX_t^\varepsilon = (\mu_t + \sigma_t \varphi_t^\varepsilon - k_t \ln X_t^\varepsilon) X_t^\varepsilon dt + \sigma_t \varepsilon^\alpha X_t^\varepsilon dB_t. \tag{3.4}$$

In reversion, it follows from (2.3) that the solution X_t^ε of (3.4) will solve (3.3) if it belongs to $\mathbf{D}_B^{1,2}$.

In the remain of this paper, we always assume that the coefficients μ_t, k_t are continuous functions and σ_t is a continuously differentiable function in $[0,T]$.

The stochastic process φ_t^ε is not bounded. However, the existence of the solution of (3.4) can be proved as in Theorem 3.1 below and we can establish the uniqueness of the solution of equation (3.4) by introducing the sequence of stopping times

$$\tau_M = \inf\{t \in [0,T] : \int_0^t (\varphi_s^\varepsilon)^2 ds > M\} \wedge T\,,$$

and considering the sequence of corresponding stopped equations

$$dX^{\varepsilon}_{t\wedge\tau_M} = (\mu_{t\wedge\tau_M} + \sigma_{t\wedge\tau_M}\varphi^{\varepsilon}_{t\wedge\tau_M} - k_{t\wedge\tau_M}\ln X^{\varepsilon}_{t\wedge\tau_M})X^{\varepsilon}_{t\wedge\tau_M}dt + \sigma_{t\wedge\tau_M}\varepsilon^{\alpha}X^{\varepsilon}_{t\wedge\tau_M}dB_t. \tag{3.5}$$

We can verify that the coefficients of (3.5) satisfy the local Lipschitz condition. Hence, the uniqueness of the solution is assured (see, for instance, [6]).

Theorem 3.1. *The solution of (3.4) is given by*

$$X^{\varepsilon}_t = \exp\Bigg(e^{-\int_0^t k_u du}\ln X_0 + \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\varepsilon^{2\alpha}\Big)e^{-\int_s^t k_u du}ds + \int_0^t \sigma_s e^{-\int_s^t k_u du}dW^{H,\varepsilon}_s\Bigg). \tag{3.6}$$

Moreover, if the Hurst index $H > \frac{1}{4}$ *then* $X^{\varepsilon}_t \in \mathbf{(H)} \subset \mathbf{D}^{1,2}_B$.

Proof. Put $Y^{\varepsilon}_t = \ln X^{\varepsilon}_t$. The Itô differential formula yields

$$dY^{\varepsilon}_t = (\mu_t + \sigma_t\varphi^{\varepsilon}_t - \frac{1}{2}\sigma_t^2\varepsilon^{2\alpha} - k_t Y^{\varepsilon}_t)dt + \sigma_t\varepsilon^{\alpha}dB_t. \tag{3.7}$$

Using a method similar to [5, Proposition 4.2], we can find the explicit solution of (3.7) by

$$Y^{\varepsilon}_t = e^{-\int_0^t k_u du}\Bigg(Y_0 + \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\varepsilon^{2\alpha}\Big)e^{\int_0^s k_u du}ds + \int_0^t \sigma_s e^{\int_0^s k_u du}dW^{H,\varepsilon}_s\Bigg).$$

Consequently, (3.6) is proved. Next, we prove that any order moment of X^{ε}_t to be finite. We have

$$E|X^{\varepsilon}_t|^n = Ee^{nY^{\varepsilon}_t} = e^{na^{\varepsilon}_t}Ee^{nm_t b^{\varepsilon}_t}, \tag{3.8}$$

where $m_t = e^{-\int_0^t k_u du}$, $a^{\varepsilon}_t = m_t\big(\ln X_0 + \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2\varepsilon^{2\alpha})m_s^{-1}ds\big)$, $b^{\varepsilon}_t = \int_0^t \sigma_s m_s^{-1}dW^{H,\varepsilon}_s$.

Since $\sigma_s m_s^{-1} \in C^1[0,T]$, the stochastic integral b^{ε}_t can be understood as Riemann-Stieltjes and as a consequence b^{ε}_t is a centered Gaussian process. We can prove its variance is finite and then $E|X_t^\varepsilon|^n$ so does for every fixed $\varepsilon > 0$. Indeed,

$$E|b_t^\varepsilon|^2 = E\Big|\int_0^t \sigma_s e^{\int_0^s k_u du}(\varphi_s^\varepsilon ds + \varepsilon^\alpha dB_s)\Big|^2$$

$$\leq 2ME\Big|\int_0^t \varphi_s^\varepsilon ds\Big|^2 + 2M\int_0^t \varepsilon^{2\alpha} ds = 2M(E|W_t^{H,\varepsilon} - \varepsilon^\alpha B_t|^2 + \int_0^t \varepsilon^{2\alpha} ds),$$

where $M = \sup\limits_{0\leq s\leq t} \sigma_s^2 e^{2\int_0^s k_u du}$.

$$E|b_t^\varepsilon|^2 \leq 2M(2E|W_t^{H,\varepsilon}|^2 + 3\varepsilon^{2\alpha} t) \leq 2M\Big[\frac{(T+\varepsilon)^{2H}}{H} + 3\varepsilon^{2\alpha} T\Big].$$

Finally, we will show that X_t^ε satisfies hypothesis **(H)**. By the Lagrange's theorem and the Hölder's inequality we have

$$\begin{aligned} E|X_t^\varepsilon - X_s^\varepsilon|^2 &= E|e^{Y_t^\varepsilon} - e^{Y_s^\varepsilon}|^2 \\ &\leq E|A_\varepsilon(t,s)(Y_t^\varepsilon - Y_s^\varepsilon)|^2 \\ &\leq [E|A_\varepsilon(t,s)|^4]^{\frac{1}{2}}[E|Y_t^\varepsilon - Y_s^\varepsilon|^4]^{\frac{1}{2}}, \end{aligned}$$

where $A_\varepsilon(t,s) = \sup\limits_{\min(Y_t^\varepsilon, Y_s^\varepsilon)\leq x\leq \max(Y_t^\varepsilon, Y_s^\varepsilon)} e^x$ that has fourth moment being finite because $A_\varepsilon(t,s) \leq e^{|Y_t^\varepsilon| + |Y_s^\varepsilon|}$.

By the inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for any $p \geq 1$ we have

$$E|Y_t^\varepsilon - Y_s^\varepsilon|^4 \leq 8\big(E|a_t^\varepsilon - a_s^\varepsilon|^4 + E|m_t b_t^\varepsilon - m_s b_s^\varepsilon|^4\big).$$

It is obvious that $|a_t^\varepsilon - a_s^\varepsilon| + |m_t - m_s| \leq M_\varepsilon|t-s|$ with some finite positive constant M_ε. We put $c_s = \sigma_s m_s^{-1} = \sigma_s e^{\int_0^s k_u du}$ then c_s is a deterministic function fulfilling hypothesis **(H)** with $\beta = \frac{1}{2}$ and b_t^ε can be represented as

$$b_t^\varepsilon = \int_0^t c_s K(t+\varepsilon,s)\,dB_s + \int_0^t \int_s^t (c_u - c_s)\,\partial_1 K(u+\varepsilon,s) du\delta B_s \tag{3.9}$$

$$= \int_0^t c_t K(t+\varepsilon,v)\,dB_v + \int_0^t \int_v^t K(u+\varepsilon,v) c_u' du dB_v$$

$$= c_t W_t^{H,\varepsilon} + \int_0^t \int_0^u K(u+\varepsilon,v) c_u' dB_v du = c_t W_t^{H,\varepsilon} + \int_0^t W_u^{H,\varepsilon} c_u' du.$$

Hence,

$$E|b_t^\varepsilon - b_s^\varepsilon|^4 \le E|c_t W_t^{H,\varepsilon} - c_s W_s^{H,\varepsilon}|^4 + \int_s^t E|W_u^{H,\varepsilon} c_u'|^4 du$$
$$\le M_\varepsilon |t-s|,$$

where M_ε is some finite positive constant. Thus, for $s,t \in [0,T]$ then

$$E|X_t^\varepsilon - X_s^\varepsilon|^2 \le M_\varepsilon |t-s|^{\frac{1}{2}}. \tag{3.10}$$

We have from the chain rule for Malliavin derivative and the expression (3.9) for any $0 \le r \le t$

$$D_r^B X_t^\varepsilon = X_t^\varepsilon D_r^B Y_t^\varepsilon = X_t^\varepsilon m_t D_r^B b_t^\varepsilon$$
$$= X_t^\varepsilon m_t \big(c_t K(t+\varepsilon,r) + \int_0^t K(u+\varepsilon,r) c_u' du \big).$$

and

$$\int_0^T E|D_r^B X_t^\varepsilon - D_r^B X_s^\varepsilon|^2 dr \tag{3.11}$$
$$\le 2 \int_0^{t\wedge s} E|X_t^\varepsilon m_t c_t K(t+\varepsilon,r) - X_s^\varepsilon m_s c_s K(s+\varepsilon,r)|^2 dr$$
$$+ 2 \int_0^{t\wedge s} E|X_t^\varepsilon m_t \int_0^t K(u+\varepsilon,r) c_u' du - X_s^\varepsilon m_s \int_0^s K(u+\varepsilon,r) c_u' du|^2 dr$$
$$\le M_\varepsilon |t-s|^{\min(\frac{1}{2},2H)},$$

where, in above estimates, we used an elementary result that

$$\int_0^{t\wedge s} |K(t+\varepsilon,r)-K(s+\varepsilon,r)|^2 dr \le E|W^H_{t+\varepsilon}-W^H_{s+\varepsilon}|^2 = |t-s|^{2H}.$$

Combining (3.10) and (3.11) we get

$$\|X^\varepsilon\|^2_{L^{1,2}_\beta} = M_\varepsilon \sup_{0<s<u<T} |t-s|^{\min(\frac{1}{2},2H)-2\beta},$$

which means that X^ε satisfies the condition (i) in **(H)** for any $\frac{1}{2}-H<\beta\le\min(\frac{1}{4},H)$, provided that $H>\frac{1}{4}$. The condition (ii) in **(H)** is proved as follows:

$$\left(\sup_{0\le t\le T} X^\varepsilon_t\right)^p \le \sup_{0\le t\le T} e^{pa^\varepsilon_t} \exp\left(\sup_{0\le t\le T} pm_t b^\varepsilon_t\right).$$

Noting that a^ε_t, m_t are deterministic functions and b^ε_t is a Gaussian process with finite variance. Therefore, $\{Z_t=\frac{pm_tb^\varepsilon_t}{\lambda}, 0\le t\le T\}$ is a pre-Gaussian process for any $\lambda>0$. By the results in [1] (Corollary 1.1, page 79 and Lemma 3.1, page 140) we have $\sup\limits_{0\le t\le T} X^\varepsilon_t \in L^p(\Omega)$ for any $p\ge\frac{1}{H}>1$.

The proof of Theorem is thus complete. □

Now taking the limit in $L^2(\Omega)$ as $\varepsilon\to 0$ we obtain the following theorem

Theorem 3.2. *Consider the fractional stochastic geometric mean reversion equation (1.2). If $H>\frac{1}{2}$, its solution is unique and given by*

$$X_t=\exp\left(e^{-\int_0^t k_u du}\ln X_0+\int_0^t \mu_s e^{-\int_s^t k_u du} ds+\int_0^t \sigma_s e^{-\int_s^t k_u du} dW^H_s\right).$$

If $H=\frac{1}{2}$, the solution is a well-known classical lognormal process

$$X_t=\exp\left(e^{-\int_0^t k_u du}\ln X_0+\int_0^t (\mu_s-\frac{1}{2}\sigma_s^2) e^{-\int_s^t k_u du} ds+\int_0^t \sigma_s e^{-\int_s^t k_u du} dB_s\right).$$

If $\frac{1}{4}<H<\frac{1}{2}$, (1.2) has no solution.

Proof. In the case, $H=\frac{1}{2}$, the proof is trivial. When $H>\frac{1}{2}$, using similar estimates as above we can prove that X_t satisfy the hypothesis

(H) and so it solves (1.2). The equation (1.2) has no solution when $\frac{1}{4} < H < \frac{1}{2}$ since $L^2\text{-}\lim\limits_{\varepsilon\to 0} X_t^\varepsilon = 0$. □

Remark. In the case $H > \frac{1}{2}$, the solution X_t is also a lognormal process because the fractional stochastic integral $\int\limits_0^t \sigma_s e^{-\int\limits_s^t k_u du} dW_s^H$ is a L^2-limit of the Gaussian process $\int\limits_0^t \sigma_s e^{-\int\limits_s^t k_u du} dW_s^{H,\varepsilon}$. This significant property make X_t as a natural candidate not only to model spot freight rate in shipping but also to model stock price in mathematical finance. We refer to [3] for an excellent application of fBm to finance.

4. Conclusion and Possible extension

The semimartingale approximate method presented in this paper can be used to study a wider class of the fractional stochastic differential equations of the form

$$dX_t = b(t, X_t)dt + \sigma_t X_t dW_t^H \ , \ 0 \le t \le T. \tag{4.1}$$

From practical point of view, it is important to find the explicit expression for the solution of each specific model. The present paper show again that the semimartingale approximate method has more advantages for this.

It is well known that in the special case $H = \frac{1}{2}$, the anticipate differential equation (4.1) is the widest class that it can be explicitly solved. In our context, (4.1) can be approximated by a classical stochastic differential equation with the same initial condition

$$dX_t^\varepsilon = [b(t, X_t^\varepsilon) + \sigma_t \varphi_t^\varepsilon X_t^\varepsilon]dt + \varepsilon^\alpha \sigma_t X_t^\varepsilon dB_t \ , \ 0 \le t \le T. \tag{4.2}$$

The explicit solution of (4.2) is given by

$$X_t^\varepsilon = \frac{Z_t^\varepsilon}{Y_t^\varepsilon},$$

where $Y_t^\varepsilon = \exp\big(\frac{1}{2}\int\limits_0^t \sigma_s^2 \varepsilon^{2\alpha} ds - \int\limits_0^t \sigma_s dW_s^{H,\varepsilon}\big)$ and Z_t^ε is the solution of the ordinary differential equation

$$dZ_t^\varepsilon = Y_t^\varepsilon b(t, \frac{Z_t^\varepsilon}{Y_t^\varepsilon})dt \ , \ Z_0^\varepsilon = X_0^\varepsilon = X_0.$$

For a suitable function $b(t,x)$, the solution of (4.1) will be a L^2-limit of X_t^ε as $\varepsilon \to 0$.

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