

# FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO FINANCE

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ABSTRACT. In this paper we use a definition of the fractional stochastic integral given by Carmona et al. (2003) in [3] and develop a simple approximation method to study quasi-linear stochastic differential equations by fractional Brownian motion. We also propose a stochastic process, namely fractional semimartingale, to model for the noise driving in some financial models.

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## 1. INTRODUCTION

A fractional Brownian motion (fBm in sort) with Hurst index  $H \in (0, 1)$  is a centered Gaussian process defined by

$$B_t^H = \int_0^t \bar{K}(t, s) dW_s, \quad (1.1)$$

where  $W$  is a standard Brownian motion and the kernel  $\bar{K}(t, s), t \geq s$ , is given by

$$\bar{K}(t, s) = C_H \left[ \frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) \int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}} (u-s)^{H-\frac{1}{2}} du \right],$$

where  $C_H = \sqrt{\frac{\pi H(2H-1)}{\Gamma(2-2H)\Gamma(H+\frac{1}{2})^2 \sin(\pi(H-\frac{1}{2}))}}$ .

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In [20] Mandelbrot and Van Ness have given a representation of  $B_t^H$  of the form:

$$B_t^H = \frac{1}{\Gamma(1 + \alpha)} \left( Z_t + \int_0^t (t - s)^\alpha dW_s \right),$$

where  $\alpha = H - \frac{1}{2}$ ,  $Z_t$  is a stochastic process of absolutely continuous trajectories, and  $W_t^H := \int_0^t (t - s)^\alpha dW_s$  is called a Liouville fBm. A Liouville fBm shares many properties of a fBm except that it has non-stationary increments (for example, see [17]). Moreover, Comte and Renault in [5] have given an excellent application of Liouville fBm to finance. Because of these reasons and for simplicity we use  $W_t^H$  throughout this paper.

The main difficulty in studying fractional stochastic calculus is that we cannot apply stochastic calculus developed by Itô since fBm is neither a Markov process nor a semimartingale, except for  $H = \frac{1}{2}$ . Recently, there have been numerous attempts to define a stochastic integral with respect to fBm. The main approaches are the following ones (refer [6, 21] for a detailed survey).

- The pathwise approach was introduced by Lin [18]. Since the trajectories of the fBm are  $\beta$ -Hölder continuous,  $\beta < H$  and by the work of Young in [31], the pathwise Riemann-Stieltjes integral exists for any integrand with sample paths  $\gamma$ -Hölder continuous with  $\gamma + H > 1$ . In [24] Nualart and Răşcanu have studied stochastic differential equations with respect to fBm with Hurst index  $H > 1/2$ .
- The regularization approach was introduced by Russo and Vallois in [27, 28] and further developed by Cheridito and Nualart in [4]. This approach has been used by Nourdin in [22] to prove the existence and uniqueness for stochastic differential equations and an approximation scheme for all  $H \in (0, 1)$ .
- The Malliavin calculus approach was introduced by Decreasefond and Üstünel in [7] for fBm and extended to more general Gaussian processes by Alos, Mazet, and Nualart in [1], Decreasefond in [8], etc. This approach leads to some different definitions for the fractional stochastic integrals such as the divergence integral, the Skorohod integral and the Stratonovich integral. By using the Skorohod integral some stochastic differential equations have been studied in [16, 23] for the case of linear equations and in [15] for the case of quasi-linear equations.

It is known that the study of the stochastic differential equations (SDEs) depends on the definitions of the stochastic integrals involved. One of the definitions of the fractional stochastic integrals is given by Carmona, Coutin and Montseny in [3]. This kind of fractional stochastic integral belongs to the third approach mentioned above and turns out to be equal to the divergence integral plus a complementary term (see Remark 18 in [6]). Thus it can be considered as a new definition of fractional stochastic integrals and naturally, the theory of SDEs needs studying independently.

In this paper we use Carmona, Coutin and Montseny's definition to study the SDEs driven by fractional Brownian motion. When the integrand is deterministic, our fractional stochastic integral coincides with the Wiener integral and SDEs of the form

$$dX_t = b(t, X_t)dt + \sigma(t)dW_t^H, \quad X_0 = x_0, \quad (1.2)$$

have been studied by Mishura (Section 3.5 in [21]). As a new contribution to (1.2), we will point out a way to find explicitly its solution.

More generally, our work deals with the following form of SDEs:

$$dX_t = b(t, X_t)dt + \sigma(t)X_t dW_t^H, \quad X_0 = x_0. \quad (1.3)$$

In order to prove the existence and uniqueness of the solution of equation (1.3) we make the following standard assumptions on coefficients: The volatility  $\sigma : [0, T] \rightarrow \mathbb{R}$  is a deterministic function on  $[0, T]$ , bounded by a constant  $M$  and the drift coefficient  $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function in all their arguments and satisfies the following conditions, for a positive constant  $L_0$  :

(C<sub>1</sub>).  $b(t, x)$  is a continuously differentiable function in  $x$  and

$$|b(t, x) - b(t, y)| \leq L_0|x - y| \quad (1.4)$$

for all  $x, y \in \mathbb{R}, t \in [0, T]$ ;

(C<sub>2</sub>). Linear growth

$$|b(t, x)| \leq L_0(1 + |x|), \quad \forall x \in \mathbb{R}, \quad \forall t \in [0, T]. \quad (1.5)$$

Sine the equation (1.3) is an anticipate SDE, similar to the Brownian case, the traditional methods cannot be applied. A method of approximation equations has been introduced recently by Tran Hung Thao (see [30] and the references therein) to solve simple linear SDEs and then used by N.T. Dung to solve more complicated SDEs such

as the fractional SDEs with polynomial drift [11] and the fractional geometric mean-reversion equations [13]. We continue to develop this method in current work with the main idea being that the existence and uniqueness of the solution for the equation (1.3) can be proved via an "approximation" equation, which is driven by semimartingales, and that the limit in  $L^2(\Omega)$  of approximation solution will be the solution of (1.3). Thus, advantages of this method are that we can still use classical Itô calculus and do not need any other fractional stochastic calculus. Based on our obtained approximation results, we propose a stochastic process, namely fractional semimartingale, to model for noise driving in some financial models.

This paper is organized as follows: In Section 2, we recall the definition of fractional stochastic integral given in [3] and some moment inequalities for fractional stochastic integral of deterministic integrands. Section 3 contains the main result of this paper which proved the existence and uniqueness of the solution of the equation (1.3). In Section 4, the European option pricing formula in the fractional semimartingale Black-Scholes model is found and the optimal portfolio in a stochastic drift model is investigated.

## 2. PRELIMINARIES

Let us recall some elements of stochastic calculus of variations.

For  $h \in L^2([0, T], \mathbb{R})$ , we denote by  $W(h)$  the Wiener integral

$$W(h) = \int_0^T h(t) dW_t.$$

Let  $\mathcal{S}$  denote the dense subset of  $L^2(\Omega, \mathcal{F}, P)$  consisting of those classes of random variables of the form

$$F = f(W(h_1), \dots, W(h_n)), \quad (2.1)$$

where  $n \in \mathbb{N}$ ,  $f \in C_b^\infty(\mathbb{R}^n, L^2([0, T], \mathbb{R}))$ ,  $h_1, \dots, h_n \in L^2([0, T], \mathbb{R})$ . If  $F$  has the form (2.1), we define its derivative as the process  $D^W F := \{D_t^W F, t \in [0, T]\}$  given by

$$D_t^W F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(W(h_1), \dots, W(h_n)) h_k(t).$$

For any  $1 \leq p < \infty$ , we shall denote by  $\mathbb{D}_W^{1,p}$  the closure of  $\mathcal{S}$  with respect to the norm

$$\|F\|_{1,p} := [E|F|^p]^{\frac{1}{p}} + E \left[ \int_0^T |D_u^W F|^p du \right]^{\frac{1}{p}},$$

and then  $\mathbb{D}_W^{1,\infty} = \bigcap_{p \geq 1} \mathbb{D}_W^{1,p}$ .

For every  $\varepsilon > 0$  we define

$$W_t^{H,\varepsilon} = \int_0^t K(t + \varepsilon, s) dW_s,$$

where  $K(t, s) = (t - s)^\alpha$ .

It is well known from [30, 12] that  $W_t^{H,\varepsilon}$  is a semimartingale with the following decomposition

$$W_t^{H,\varepsilon} = \varepsilon^\alpha W_t + \int_0^t \varphi_s^\varepsilon ds, \quad (2.2)$$

where  $\varphi_s^\varepsilon = \int_0^s \alpha(s + \varepsilon - u)^{\alpha-1} dW_u$ . Moreover,  $W_t^{H,\varepsilon}$  converges in  $L^p(\Omega)$ ,  $p > 1$  uniformly in  $t \in [0, T]$  to  $W_t^H$  as  $\varepsilon \rightarrow 0$ :

$$E|W_t^{H,\varepsilon} - W_t^H|^p \leq c_p \varepsilon^{pH}.$$

It is well known from [3, 6] that for an adapted process  $f$  belonging to the space  $\mathbb{D}_W^{1,2}$  we have

$$\begin{aligned} \int_0^t f_s dW_s^{H,\varepsilon} &= \int_0^t f_s K(s + \varepsilon, s) dW_s + \int_0^t f_s \varphi_s^\varepsilon ds \\ &= \int_0^t f_s K(t + \varepsilon, s) dW_s + \int_0^t \int_s^t (f_u - f_s) \partial_1 K(u + \varepsilon, s) du \delta W_s \\ &\quad + \int_0^t \int_0^u D_s^W f_u \partial_1 K(u + \varepsilon, s) ds du, \quad (2.3) \end{aligned}$$

where  $\partial_1 K(u, s) = \frac{\partial}{\partial u} K(u, s)$  and the second integral in the right-hand side is a Skorokhod integral (we refer the reader to [25] for more details about the Skorokhod integral).

**Hypothesis (H):** Assume that  $f$  is an adapted process belonging to the space  $\mathbb{D}_W^{1,2}$  and that there exists  $\beta$  fulfilling  $\beta + H > 1/2$  and  $p > 1/H$  such that

- (i)  $\|f\|_{L_\beta^{1,2}}^2 := \sup_{0 < s < u < T} \frac{E[(f_u - f_s)^2 + \int_0^T (D_r^W f_u - D_r^W f_s)^2 dr]}{|u-s|^{2\beta}}$  is finite,
- (ii)  $\sup_{0 < s < T} |f_s|$  belongs to  $L^p(\Omega)$ .

For  $f \in (\mathbf{H})$ ,  $\int_0^t f_s dW_s^{H,\varepsilon}$  converges in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$ . Each term in the right-hand side of (2.3) converges to the same term where  $\varepsilon = 0$ . Then, it is "natural" to define

**Definition 2.1.** Let  $f \in (\mathbf{H})$ . The fractional stochastic integral of  $f$  with respect to  $W^H$  is defined by

$$\begin{aligned} \int_0^t f_s dW_s^H &= \int_0^t f_s K(t, s) dW_s + \int_0^t \int_s^t (f_u - f_s) \partial_1 K(u, s) du \delta W_s \\ &\quad + \int_0^t du \int_0^u D_s^W f_u \partial_1 K(u, s) ds, \end{aligned} \quad (2.4)$$

where  $K(t, s) = (t - s)^\alpha$ ,  $\partial_1 K(t, s) = \alpha(t - s)^{\alpha-1}$ .

If  $f$  is a deterministic function in  $L^{\frac{1}{H}}[0, T]$  then our fractional stochastic integral coincides with the Wiener integral with respect to fBm. Indeed, we have

$$I_t := \int_0^t f_s dW_s^H = \alpha \int_0^t \int_s^t f_u (u - s)^{\alpha-1} du dW_s. \quad (2.5)$$

**Proposition 2.1.** Suppose that  $H > \frac{1}{2}$ . Let  $f \in L^{\frac{1}{H}}[0, T]$  then for any  $p > 0$ , there exists a constant  $C(p, H)$  such that

$$E|I_T^*|^p \leq C(p, H) \|f\|_{L^{\frac{1}{H}}[0, T]}^p, \quad (2.6)$$

where  $I_T^* = \sup_{0 \leq t \leq T} |I_t|$ . Consequently,

$$E \exp(I_T^*) \leq 2 \exp \left( 4\sqrt{2} \bar{C}_H \|f\|_{L^{\frac{1}{H}}[0,T]} + \frac{1}{2} C(2, H) \|f\|_{L^{\frac{1}{H}}[0,T]}^2 \right), \quad (2.7)$$

where  $\bar{C}_H = \frac{1}{2} (C(2, H))^{\frac{1}{2}} H \int_{\ln 2}^{\infty} z^{\frac{1}{2}} \frac{e^z dz}{(e^z - 1)^{H+1}}$ .

*Proof.* The inequality (2.6) was proved by Mishura in Theorem 1.10.3 [21]. Denote  $\gamma^2 = \sup_{0 \leq t \leq T} E[I_t^2]$ , then for any  $r > 4\sqrt{2}D(T, \frac{\gamma}{2})$  we have

$$P(I_T^* > r) \leq 2 \left( 1 - \Phi \left( \frac{r - 4\sqrt{2}D(T, \frac{\gamma}{2})}{\gamma} \right) \right), \quad (2.8)$$

where  $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$  and  $D(T, \frac{\gamma}{2})$  is the Dudley integral (see, [10]).

$$\begin{aligned} E \exp(I_T^*) &= 1 + \int_0^{\infty} e^x P(I_T^* > x) dx \leq 1 + \int_0^{4\sqrt{2}D(T, \frac{\gamma}{2})} e^x P(I_T^* > x) dx \\ &\quad + \int_{4\sqrt{2}D(T, \frac{\gamma}{2})}^{\infty} 2e^x \left( 1 - \Phi \left( \frac{x - 4\sqrt{2}D(T, \frac{\gamma}{2})}{\gamma} \right) \right) dx \\ &\leq e^{4\sqrt{2}D(T, \frac{\gamma}{2})} + 2 \int_0^{\infty} e^{x+4\sqrt{2}D(T, \frac{\gamma}{2})} \left( 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\gamma} e^{-\frac{y^2}{2}} dy \right) dx \\ &\leq e^{4\sqrt{2}D(T, \frac{\gamma}{2})} + 2e^{4\sqrt{2}D(T, \frac{\gamma}{2})} \int_0^{\infty} e^x \left( \frac{1}{\sqrt{2\pi}} \int_{x/\gamma}^{\infty} e^{-\frac{y^2}{2}} dy \right) dx \\ &\leq e^{4\sqrt{2}D(T, \frac{\gamma}{2})} + 2e^{4\sqrt{2}D(T, \frac{\gamma}{2})} \left( e^{\frac{\gamma^2}{2}} - \frac{1}{2} \right) = 2e^{4\sqrt{2}D(T, \frac{\gamma}{2}) + \frac{\gamma^2}{2}}. \end{aligned}$$

Evidently,

$$\gamma^2 \leq C(2, H) \|f\|_{L^{\frac{1}{H}}[0,T]}^2.$$

Moreover, from the proof of Theorem 1.10.3 in [21] we know that

$$D(T, \frac{\gamma}{2}) \leq \bar{C}_H \|f\|_{L^{\frac{1}{H}}[0,T]},$$

Thus the proposition is proved.  $\square$

**Proposition 2.2.** *Suppose that  $H \in (0, 1)$ . Let  $f$  be a bounded deterministic function on  $[0, T] : |f(t)| \leq M \forall t \in [0, T]$ . Then the integral process  $I_t \in C^{H^-}[0, T] = \bigcap_{0 < h < H} C^h[0, T]$  a.s.*

*Proof.* Since  $I_t$  is a Gaussian process,  $I_t \in C^{H^-}[0, T]$  follows from the following inequality

$$\begin{aligned}
E|I_t - I_r|^p &\leq C_p \left( \int_0^r \left( \int_r^t \frac{\alpha f_u}{(u-s)^{1-\alpha}} du \right)^2 ds \right)^{\frac{p}{2}} \\
&\quad + C_p \left( \int_r^t \left( \int_s^t \frac{\alpha f_u}{(u-s)^{1-\alpha}} du \right)^2 ds \right)^{\frac{p}{2}} \\
&\leq C_p \left( \int_0^r \left( \int_r^t \frac{\alpha M}{(u-s)^{1-\alpha}} du \right)^2 ds \right)^{\frac{p}{2}} + C_p \left( \int_r^t \left( \int_s^t \frac{\alpha M}{(u-s)^{1-\alpha}} du \right)^2 ds \right)^{\frac{p}{2}} \\
&\leq C_p M^p \left( \int_0^r [(t-s)^\alpha - (r-s)^\alpha]^2 ds \right)^{\frac{p}{2}} + C_p M^p \left( \int_r^t (t-s)^{2\alpha} ds \right)^{\frac{p}{2}} \\
&= C_p M^p (E|W_t^H - W_r^H|^2)^{\frac{p}{2}} + C_p M^p \left( \frac{(t-r)^{2H}}{2H} \right)^{\frac{p}{2}} \\
&\leq C_p M^p \left( 1 + \frac{1}{(2H)^{p/2}} \right) |t-r|^{pH},
\end{aligned}$$

where  $C_p$  is a finite constant. In the above inequalities we used the Burkholder-Davis-Gundy inequality and fundamental inequality  $(a+b)^p \leq c_p(a^p + b^p)$ , where  $c_p = 1$  if  $0 < p \leq 1$  and  $c_p = 2^{p-1}$  if  $p > 1$ .  $\square$

### 3. FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

In this whole section we consider only  $H > \frac{1}{2}$ . Since the Malliavin derivative  $D_u^W f_s = 0$  for any deterministic function  $f_s$ , we have the following definition.



**Definition 3.1.** The solution of (1.3) is a stochastic process  $X_t$  belonging to the space  $(\mathbf{H})$  and has a form for all  $t \in [0, T]$

$$\begin{aligned} X_t = X_0 &+ \int_0^t b(s, X_s) ds + \int_0^t \sigma(s) X_s K(t, s) dW_s \\ &+ \int_0^t \int_s^t (\sigma(u) X_u - \sigma(s) X_s) \partial_1 K(u, s) du \delta W_s \\ &+ \int_0^t \int_0^s \sigma(s) D_u^W X_s \partial_1 K(s, u) dud s. \end{aligned} \quad (3.1)$$

We can see that (3.1) contains the Skorokhod integral and the Malliavin derivative, so we cannot apply standard methods (for instance, Picard iteration procedure) to prove the existence and uniqueness of the solution. However, Definition 2.1 for the fractional stochastic integral leads us to consider the "approximation" equation corresponding to (3.1)

$$\begin{aligned} X_t^\varepsilon = X_0 &+ \int_0^t b(s, X_s^\varepsilon) ds + \int_0^t \sigma(s) X_s^\varepsilon K(t + \varepsilon, s) dW_s \\ &+ \int_0^t \int_s^t (\sigma(u) X_u^\varepsilon - \sigma(s) X_s^\varepsilon) \partial_1 K(u + \varepsilon, s) du \delta W_s \\ &+ \int_0^t \int_0^s \sigma(s) D_u^W X_s^\varepsilon \partial_1 K(s + \varepsilon, u) dud s, \end{aligned} \quad (3.2)$$

which can be reduced to

$$dX_t^\varepsilon = b(t, X_t^\varepsilon) dt + \sigma(t) X_t^\varepsilon dW_t^{H, \varepsilon}, \quad X_0^\varepsilon = x_0. \quad (3.3)$$

Note that the hypothesis  $(\mathbf{H})$  ensures the convergence of approximation integrals to the fractional stochastic integrals. Hence, if both  $X_t^\varepsilon$  and its  $L^2$ -lim  $X_t^\varepsilon$  belong to  $(\mathbf{H})$  then we can take the limit of (3.2) as  $\varepsilon \rightarrow 0$  to get (3.1). Thus  $L^2$ -lim  $X_t^\varepsilon$  will be the solution of (3.1).

**Theorem 3.1.** Fix  $\varepsilon > 0$ . The equation (3.3) has a unique solution  $X_t^\varepsilon$  satisfying the hypothesis  $(\mathbf{H})$  and converging in  $L^2(\Omega)$  when  $\varepsilon \rightarrow 0$ .

*Proof. Existence and Uniqueness:* By (2.2) we can rewrite (3.3) as a classical Itô equation

$$dX_t^\varepsilon = [b(t, X_t^\varepsilon) + \varphi_t^\varepsilon \sigma(t) X_t^\varepsilon] dt + \varepsilon^\alpha \sigma(t) X_t^\varepsilon dW_t, \quad 0 \leq t \leq T. \quad (3.4)$$

As a consequence, the explicit solution of (3.3) is given by

$$X_t^\varepsilon = \frac{Z_t^\varepsilon}{Y_t^\varepsilon},$$

where  $Y_t^\varepsilon = \exp\left(\frac{1}{2}\varepsilon^{2\alpha} \int_0^t \sigma^2(s) ds - \int_0^t \sigma(s) dW_s^{H,\varepsilon}\right) \in \bigcap_{0 < \gamma < \frac{1}{2}} C^\gamma[0, T]$  a.s.

and  $Z_t^\varepsilon$  is the solution of the ordinary differential equation

$$dZ_t^\varepsilon = Y_t^\varepsilon b\left(t, \frac{Z_t^\varepsilon}{Y_t^\varepsilon}\right) dt, \quad Z_0^\varepsilon = X_0^\varepsilon = x_0. \quad (3.5)$$

For  $P$ -almost all  $\omega \in \Omega$ , the function  $f^\varepsilon(t, x, \omega) = Y_t^\varepsilon b\left(t, \frac{x}{Y_t^\varepsilon}\right)$  satisfies global Lipschitzian condition in  $x \in \mathbb{R}$ . Hence, the equation (3.5) has a unique solution  $Z_t^\varepsilon \in C^1[0, T]$  a.s., so (3.3) does.

**Hypothesis (H):** First, we show that  $X_t^\varepsilon \in \mathbb{D}_W^{1,2}$  by using Theorem 2.2.1 in [25]. Since the stochastic process  $\varphi_t^\varepsilon$  is not bounded we need to introduce the increasing sequence of stopping times

$$\tau_M = \inf\left\{t \in [0, T] : \int_0^t (\varphi_s^\varepsilon)^2 ds > M\right\} \wedge T,$$

and consider the sequence of stopped equations corresponding to (3.4)

$$\begin{aligned} dX_{t \wedge \tau_M}^\varepsilon &= [b(t \wedge \tau_M, X_{t \wedge \tau_M}^\varepsilon) + \varphi_{t \wedge \tau_M}^\varepsilon \sigma(t \wedge \tau_M) X_{t \wedge \tau_M}^\varepsilon] dt \\ &\quad + \varepsilon^\alpha \sigma(t \wedge \tau_M) X_{t \wedge \tau_M}^\varepsilon dW_t, \quad 0 \leq t \leq T. \end{aligned} \quad (3.6)$$

It is easy to see that the coefficients of (3.6) are globally Lipschitz functions with linear growth. Hence,  $X_{t \wedge \tau_M}^\varepsilon \in \mathbb{D}_W^{1,\infty} \subset \mathbb{D}_W^{1,2}$ , and then by taking limit  $M \rightarrow \infty$  and by closability of the Malliavin derivative we obtain  $X_t^\varepsilon \in \mathbb{D}_W^{1,\infty}$ . Moreover,  $b(t, x)$  is a continuously differentiable in  $x$ , from Theorem 2.2.1 in [25] we know that  $D_s^W X_t^\varepsilon$  solves the following linear equation

$$\begin{aligned} D_s^W X_t^\varepsilon &= \varepsilon^\alpha \sigma(s) X_s^\varepsilon + \int_s^t [(b'_x(r, X_r^\varepsilon) + \sigma(r) \varphi_r^\varepsilon) D_s^W X_r^\varepsilon \\ &\quad + \alpha \sigma(r) X_r^\varepsilon (r - s + \varepsilon)^{\alpha-1}] dr + \varepsilon^\alpha \int_s^t \sigma(r) D_s^W X_r dW_r \end{aligned} \quad (3.7)$$

for  $s \leq t$ .

The explicit solution of the above equation is given by

$$D_s^W X_t^\varepsilon = \Theta_{s,t}^\varepsilon \left( \varepsilon^\alpha \sigma(s) X_s^\varepsilon + \int_s^t \alpha \sigma(r) X_r^\varepsilon (r-s+\varepsilon)^{\alpha-1} (\Theta_{s,r}^\varepsilon)^{-1} dr \right) \mathbf{1}_{\{s \leq t\}},$$

or equivalently

$$D_s^W X_t^\varepsilon = \left( \varepsilon^\alpha \sigma(s) X_s^\varepsilon \Theta_{s,t}^\varepsilon + \int_s^t \alpha \sigma(r) X_r^\varepsilon (r-s+\varepsilon)^{\alpha-1} \Theta_{r,t}^\varepsilon dr \right) \mathbf{1}_{\{s \leq t\}},$$

where

$$\Theta_{s,t}^\varepsilon = \exp \left( \int_s^t [b'_x(r, X_r^\varepsilon) - \frac{1}{2} \varepsilon^{2\alpha} \sigma^2(r)] dr + \int_s^t \sigma(r) dW_r^{H,\varepsilon} \right).$$

Next, we prove that  $X_t^\varepsilon$  satisfies the condition (ii): Put

$$I_t^{(\varepsilon)} := \int_0^t \sigma(s) dW_s^{H,\varepsilon} = \int_0^t (\varepsilon^\alpha \sigma(s) + \int_s^t \sigma(u) \partial_1 K(u+\varepsilon, s) du) dW_s,$$

$$I_t := \int_0^t \sigma(s) dW_s^H = \int_0^t \int_s^t \sigma(u) \partial_1 K(u, s) du dW_s.$$

Then  $I_t^{(\varepsilon)} \rightarrow I_t$  uniformly in  $t \in [0, T]$  as  $\varepsilon \rightarrow 0$  since  $I_t^{(\varepsilon)}, I_t$  are Gaussian processes and

$$\begin{aligned} E |I_t^{(\varepsilon)} - I_t|^p &\leq C_p \left( \int_0^t \varepsilon^{2\alpha} \sigma^2(s) ds \right)^{p/2} \\ &+ C_p \left\{ \int_0^t \left( \int_s^t \alpha \sigma(u) [(u-s+\varepsilon)^{\alpha-1} - (u-s)^{\alpha-1}] du \right)^2 ds \right\}^{p/2} \leq C_p M^p \varepsilon^{p\alpha} \\ &+ C_p M^p \left\{ \int_0^t \left( (t-s)^\alpha - (t-s+\varepsilon)^\alpha + \varepsilon^\alpha \right)^2 ds \right\}^{p/2} \\ &\leq C_p M^p (1 + 2T^{p/2}) \varepsilon^{p\alpha}. \quad (3.8) \end{aligned}$$

Hence, from (2.7),  $E \exp(\sup_{0 \leq t \leq T} |pI_t^{(\varepsilon)}|)$  is bounded for small enough values of  $\varepsilon$ . We have for all  $p > 1$

$$\begin{aligned} E|\sup_{0 \leq t \leq T} Y_t^\varepsilon|^p &\leq E \exp \left\{ \sup_{0 \leq t \leq T} \left( \frac{p}{2} \varepsilon^{2\alpha} \int_0^t \sigma^2(s) ds - p \int_0^t \sigma(s) dW_s^{H,\varepsilon} \right) \right\} \\ &\leq \exp \left( \frac{p}{2} \varepsilon^{2\alpha} \int_0^T \sigma^2(s) ds \right) E \exp(\sup_{0 \leq t \leq T} |pI_t^{(\varepsilon)}|) \leq C_{p,H,T}^{(1)} < \infty, \end{aligned}$$

where  $C_{p,H,T}^{(1)}$  is a constant depending only on  $p, H, T$ . Similarly,

$$E|\sup_{0 \leq t \leq T} \frac{1}{Y_t^\varepsilon}|^p \leq C_{p,H,T}^{(1)}. \quad (3.9)$$

From (3.5) and by assumption (1.5) we get

$$\begin{aligned} Z_u^\varepsilon &= x_0 + \int_0^u Y_s^\varepsilon b(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}) ds, \\ \sup_{0 \leq u \leq t} |Z_u^\varepsilon| &\leq |x_0| + \int_0^t |Y_s^\varepsilon b(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon})| ds \leq |x_0| + \int_0^t \sup_{0 \leq u \leq s} |Y_u^\varepsilon b(s, \frac{Z_u^\varepsilon}{Y_u^\varepsilon})| ds, \end{aligned}$$

$$\begin{aligned} E(\sup_{0 \leq u \leq t} |Z_u^\varepsilon|)^p &\leq 2^{p-1}|x_0|^p + 2^{p-1}L_0^p \int_0^t E(\sup_{0 \leq u \leq s} |Y_u^\varepsilon + Z_u^\varepsilon|)^p ds \\ &\leq 2^{p-1}|x_0|^p + 4^{p-1}L_0^p \int_0^t [E(\sup_{0 \leq u \leq s} |Y_u^\varepsilon|)^p + E(\sup_{0 \leq u \leq s} |Z_u^\varepsilon|)^p] ds. \end{aligned}$$

An application of Gronwall's Lemma to the latest inequality yields

$$E(\sup_{0 \leq u \leq t} |Z_u^\varepsilon|)^p \leq (2^{p-1}|x_0|^p + 4^{p-1}L_0^p T C_{p,H,T}) e^{4^{p-1}L_0^p T} := C_{p,H,T}^{(2)}. \quad (3.10)$$

We now combine (3.9) and (3.10) to get

$$\begin{aligned} E(\sup_{0 \leq u \leq t} |X_t^\varepsilon|)^p &\leq \left\{ E(\sup_{0 \leq u \leq t} |Z_t^\varepsilon|)^{2p} E(\sup_{0 \leq u \leq t} |\frac{1}{Y_t^\varepsilon}|)^{2p} \right\}^{\frac{1}{2}} \\ &\leq \left\{ C_{2p,H,T}^{(1)} C_{2p,H,T}^{(2)} \right\}^{\frac{1}{2}} := C_{p,H,T}^{(3)}. \quad (3.11) \end{aligned}$$

Finally, we prove that  $X_t^\varepsilon$  satisfies the condition (i):

Since  $b'_x(t, x)$  is bounded and  $\Theta_{s,t}^\varepsilon = \exp\left(\int_s^t b'_x(r, X_r^\varepsilon)\frac{Y_r^\varepsilon}{Y_t^\varepsilon}\right)$  we get

$$E\left(\sup_{0 \leq s \leq t \leq T} \Theta_{s,t}^\varepsilon\right)^p < \infty.$$

Then using the inequality (3.12) below we can see that there exists a constant  $C_{p,H,T}^{(4)}$  depending only on  $p, H, T$  such that

$$E|\Theta_{s,t_2}^\varepsilon - \Theta_{s,t_1}^\varepsilon|^p \leq C_{p,H,T}^{(4)}|t_1 - t_2|^{p/2}, \quad \forall t_1, t_2 \in [0, T].$$

$$\begin{aligned} E|D_s^W X_{t_2}^\varepsilon - D_s^W X_{t_1}^\varepsilon|^2 &\leq E|\varepsilon^\alpha \sigma(s) X_s^\varepsilon (\Theta_{s,t_2}^\varepsilon - \Theta_{s,t_1}^\varepsilon)|^2 \\ &+ E\left|\int_s^{t_1} \alpha \sigma(r) X_r^\varepsilon (r - s + \varepsilon)^{\alpha-1} (\Theta_{s,t_2}^\varepsilon - \Theta_{s,t_1}^\varepsilon) dr\right|^2 \\ &+ E\left|\int_{t_1}^{t_2} \alpha \sigma(r) X_r^\varepsilon (r - s + \varepsilon)^{\alpha-1} \Theta_{r,t_2}^\varepsilon dr\right|^2 := A_1 + A_2 + A_3. \end{aligned}$$

It is easy to see that  $A_1 \leq C_{H,T}^{(5)}|t_1 - t_2|$ ,  $\forall t_1, t_2 \in [0, T]$ . Applying Hölder's inequality to  $p_1 = \frac{2-\alpha}{2-2\alpha} \in (1, \frac{1}{1-\alpha})$ ,  $q_1 = \frac{p_1}{p_1-1}$  we have

$$\begin{aligned} A_2 &\leq \left(\int_s^{t_1} (r - s + \varepsilon)^{p_1 \alpha - p_1} dr\right)^{\frac{2}{p_1}} E\left|\int_s^{t_1} (\alpha \sigma(r) X_r^\varepsilon (\Theta_{s,t_2}^\varepsilon - \Theta_{s,t_1}^\varepsilon))^{q_1} dr\right|^{\frac{2}{q_1}} \\ &\leq \left(\int_s^{t_1} (r - s + \varepsilon)^{p_1 \alpha - p_1} dr\right)^{\frac{2}{p_1}} \int_s^{t_1} E|\alpha \sigma(r) X_r^\varepsilon (\Theta_{s,t_2}^\varepsilon - \Theta_{s,t_1}^\varepsilon)|^2 dr \\ &\leq C_{H,T}^{(6)}|t_1 - t_2|. \end{aligned}$$

Similarly, we have also  $A_3 \leq C_{H,T}^{(7)}|t_1 - t_2|$ . Thus,

$$E|D_s^W X_{t_2}^\varepsilon - D_s^W X_{t_1}^\varepsilon|^2 \leq C_{H,T}^{(8)}|t_1 - t_2|.$$

Since  $X_t^\varepsilon = \frac{Z_t^\varepsilon}{Y_t^\varepsilon}$ , it is easy to show that

$$E|X_{t_2}^\varepsilon - X_{t_1}^\varepsilon|^2 \leq C_{H,T}^{(9)}|t_1 - t_2|.$$

Consequently, if we choose  $\beta$  such that  $\frac{1}{2} - H < \beta < \frac{1}{2}$  then

$$\|X^\varepsilon\|_{L_\beta^{1,2}}^2 \leq C_{H,T}^{(10)} \sup_{0 < t_1 < t_2 < T} |t_2 - t_1|^{1-2\beta} < \infty.$$

**Convergence:** Let  $X_1, X_2$  be two random variables. By Lagrange's theorem and Hölder's inequality we have

$$\begin{aligned} E|e^{X_1} - e^{X_2}|^p &\leq E|(X_1 - X_2) \sup_{\min(X_1, X_2) \leq x \leq \max(X_1, X_2)} e^x|^p \\ &\leq E|(X_1 - X_2)e^{|X_1|+|X_2|}|^p \leq \left( E[e^{2p|X_1|} + e^{2p|X_2|}]E|X_1 - X_2|^{2p} \right)^{\frac{1}{2}}. \end{aligned} \quad (3.12)$$

We now put  $Y_t = \exp\left(-\int_0^t \sigma(s)dW_s^H\right)$  and apply (3.12) to  $X_1 = \frac{1}{2}\varepsilon^{2\alpha} \int_0^t \sigma^2(s)ds - \int_0^t \sigma(s)dW_s^{H,\varepsilon}$ ,  $X_2 = -\int_0^t \sigma(s)dW_s^H$  to get

$$E|Y_t^\varepsilon - Y_t|^p \leq \left( E[e^{2p|X_1|} + e^{2p|X_2|}]E|X_1 - X_2|^{2p} \right)^{\frac{1}{2}}. \quad (3.13)$$

It is obvious that  $E[e^{2p|X_1|} + e^{2p|X_2|}]$  is finite because  $I_t^{(\varepsilon)}$  and  $I_t$  are centered Gaussian processes with finite variances in  $[0, T]$ . Moreover, by the fundamental inequality  $(a + b)^p \leq c_p(a^p + b^p)$ , where  $c_p = 1$  if  $0 < p \leq 1$  and  $c_p = 2^{p-1}$  if  $p > 1$

$$E|X_1 - X_2|^{2p} \leq c_{2p} \left[ E|I_t^{(\varepsilon)} - I_t|^{2p} + \frac{1}{2^{2p}} \varepsilon^{4p\alpha} \left( \int_0^t \sigma^2(s)ds \right)^{2p} \right] \leq C\varepsilon^{2p\alpha}, \quad (3.14)$$

where  $C$  is a finite constant. Consequently,  $Y_t^\varepsilon \rightarrow Y_t$  in  $L^p(\Omega)$  as  $\varepsilon \rightarrow 0$ .

We consider

$$dZ_t = Y_t b\left(t, \frac{Z_t}{Y_t}\right) dt, \quad Z_0 = x_0. \quad (3.15)$$

For  $P$ -almost all  $\omega \in \Omega$ , the function  $f(t, x, \omega) = Y_t b\left(t, \frac{x}{Y_t}\right)$  satisfies the global Lipschitzian condition in  $x \in \mathbb{R}$ . Hence, the equation (3.15) has a unique solution  $Z_t \in C^1[0, T]$ . Moreover,  $Z_t^\varepsilon \xrightarrow{L^2(\Omega)} Z_t$  uniformly in  $t \in [0, T]$ . Indeed, we have

$$\begin{aligned} E|Z_t^\varepsilon - Z_t|^2 &\leq \int_0^t E \left[ Y_s^\varepsilon b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) - Y_s b\left(s, \frac{Z_s}{Y_s}\right) \right]^2 ds \\ &\leq 2 \int_0^t E \left[ Y_s^\varepsilon b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) - Y_s b\left(s, \frac{Z_s}{Y_s}\right) \right]^2 ds \end{aligned} \quad (3.16)$$

$$\begin{aligned}
 & + 2 \int_0^t E[Y_s b(s, \frac{Z_s^\varepsilon}{Y_s}) - Y_s b(s, \frac{Z_s}{Y_s})]^2 ds \\
 & \leq 2 \int_0^t E[Y_s^\varepsilon b(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}) - Y_s b(s, \frac{Z_s^\varepsilon}{Y_s})]^2 ds + 2L_0^2 \int_0^t E|Z_s^\varepsilon - Z_s|^2 ds \\
 & \qquad \qquad \qquad := A_4 + 2L_0^2 \int_0^t E|Z_s^\varepsilon - Z_s|^2 ds.
 \end{aligned}$$

By the assumptions (1.4), (1.5) we have

$$\begin{aligned}
 |Y_s^\varepsilon b(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}) - Y_s b(s, \frac{Z_s^\varepsilon}{Y_s})| & \leq |(Y_s^\varepsilon - Y_s) b(s, \frac{Z_s^\varepsilon}{Y_s})| \\
 & + |Y_s^\varepsilon (b(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}) - b(s, \frac{Z_s^\varepsilon}{Y_s}))| \leq L_0 |(Y_s^\varepsilon - Y_s) (\frac{Z_s^\varepsilon}{Y_s} + 1)| \\
 & \qquad \qquad \qquad + L_0 |Y_t^\varepsilon (\frac{Z_s^\varepsilon}{Y_s^\varepsilon} - \frac{Z_s^\varepsilon}{Y_s})|,
 \end{aligned}$$

which means that  $A_4 \xrightarrow{L^2(\Omega)} 0$  uniformly in  $t \in [0, T]$ . As a consequence, the fact that  $Z_t^\varepsilon \xrightarrow{L^2(\Omega)} Z_t$  uniformly in  $t \in [0, T]$  follows from (3.16) and by the Gronwall's Lemma.

Now we put  $X_t = \frac{Z_t}{Y_t}$  then  $X_t^\varepsilon \xrightarrow{L^2(\Omega)} X_t$  as  $\varepsilon \rightarrow 0$ . We note also  $X_t \in C^{H^-}[0, T]$  because  $Y_t \in C^{H^-}[0, T]$  (this fact follows from Proposition 2.2) and  $Z_t \in C^1[0, T]$  a.s.

The Theorem thus is proved.  $\square$

By taking limit when  $\varepsilon \rightarrow 0$ , the main result of this section is formulated in the theorem given below.

**Theorem 3.2.** *The fractional stochastic differential equation (1.3) has a solution which is given by*

$$X_t = \frac{Z_t}{Y_t}, \tag{3.17}$$

where  $Y_t = \exp\left(-\int_0^t \sigma(s)dW_s^H\right)$  and  $Z_t$  is the unique solution of the ordinary differential equation

$$dZ_t = Y_t b\left(t, \frac{Z_t}{Y_t}\right) dt, \quad Z_0 = x_0. \quad (3.18)$$

Moreover,  $X_t \in C^{H^-}[0, T] \cap \mathbb{D}_W^{1, \infty}$  and

$$D_s^W X_t = \left( \int_s^t \alpha \sigma(r) X_r (r-s)^{\alpha-1} \Theta_{r,t} dr \right) \mathbf{1}_{\{s \leq t\}},$$

where  $\Theta_{s,t} = \exp\left(\int_s^t b'_x(r, X_r) dr + \int_s^t \sigma(r) dW_r^H\right)$ .

*Proof.* By using similar estimates as above we can see that  $X_t$  defined by (3.17) satisfies the hypothesis **(H)** and then we can take the limit of both sides of (3.2) when  $\varepsilon \rightarrow 0$ , each term in the right-hand side converges to the same term where  $\varepsilon = 0$ . So  $X_t$  solves (1.3).  $\square$

**Example:** The solution of the fractional Black-Scholes equation of the form

$$dX_t = b(t)X_t dt + \sigma(t)X_t dW_t^H, \quad X_0 = x_0,$$

is given by

$$X_t = x_0 \exp\left(\int_0^t b(s) ds + \int_0^t \sigma(s) dW_s^H\right).$$

*Remark 3.1.* We now turn our attention to the the equation (1.2) mentioned in Introduction:

$$dX_t = b(t, X_t) dt + \sigma(t) dW_t^H, \quad X_0 = x_0, \quad (3.19)$$

where  $\sigma(t)$  is a deterministic function and  $b$  is Lipschitz continuous and satisfies the linear growth condition. The existence and uniqueness of the solution of (3.19) were discussed by Mishura (see Section 3.5 in [21]).

Consider the approximation equations corresponding to (3.19).

$$dX_t^\varepsilon = b(t, X_t^\varepsilon) dt + \sigma(t) dW_t^{H, \varepsilon}, \quad X_0^\varepsilon = x_0. \quad (3.20)$$



By Gronwall's lemma it is obvious that  $X_t^\varepsilon \rightarrow X_t$  in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$  since the definition of our fractional stochastic integral and

$$E|X_t^\varepsilon - X_t|^2 \leq 2L_0^2 \int_0^t E|X_s^\varepsilon - X_s|^2 ds + 2E \left| \int_0^t \sigma(s) d(W_s^{H,\varepsilon} - W_s^H) \right|^2.$$

Thus our approximate method works for all  $H \in (0, 1)$ . This gives us the following scheme to solve (3.19):

*Step 1:* Solving (3.20) by using the classical Itô differential formula.

*Step 2:* Taking limit of the solution found in Step 1 in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$  to get the solution of (3.19).

We illustrate this scheme by finding the solution for the equation with nonlinear drift of the form

$$dX_t = (e^{cX_t} + b)dt + \sigma dW_t^H, \quad X_0 = x_0, \quad (3.21)$$

where  $b, c, \sigma, x_0$  are real constants and  $c \neq 0$ . The approximation equation corresponding to (3.21) is

$$dX_t^\varepsilon = (e^{cX_t^\varepsilon} + b)dt + \sigma dW_t^{H,\varepsilon}, \quad X_0^\varepsilon = x_0,$$

or equivalently,

$$dX_t^\varepsilon = (e^{cX_t^\varepsilon} + b + \sigma \varphi_t^\varepsilon)dt + \sigma \varepsilon^\alpha dW_t.$$

An application of the classical Itô differential formula to  $Y_t^\varepsilon := e^{-cX_t^\varepsilon}$  yields

$$Y_t^\varepsilon = \left( \left( \frac{1}{2} c^2 \sigma^2 \varepsilon^{2\alpha} - bc - c\varphi_t^\varepsilon \right) Y_t^\varepsilon - c \right) dt - c\sigma \varepsilon^\alpha Y_t^\varepsilon dW_t. \quad (3.22)$$

The equation (3.22) is a linear SDE and its solution is given by

$$Y_t^\varepsilon = \exp(-bct - \sigma c W_t^{H,\varepsilon} - cx_0) \left( 1 - \int_0^t c \exp(cx_0 + bcs + \sigma c W_s^{H,\varepsilon}) ds \right).$$

As a consequence,

$$X_t^\varepsilon = bt + \sigma W_t^{H,\varepsilon} + x_0 - \frac{1}{c} \ln \left( 1 - c \int_0^t \exp(cx_0 + bcs + \sigma c W_s^{H,\varepsilon}) ds \right),$$

and then the solution of (3.21) is

$$X_t = bt + \sigma W_t^H + x_0 - \frac{1}{c} \ln \left( 1 - c \int_0^t \exp(cx_0 + bcs + \sigma c W_s^H) ds \right).$$

#### 4. APPLICATIONS TO FINANCE

It is known that fBm with Hurst index  $H \neq \frac{1}{2}$  is a Gaussian process that has a memory. More precisely, let  $\rho_H(n) := E(W_1^H(W_{n+1}^H - W_n^H))$ , then (see, [20])

$$\rho_H(n) \approx H(2H - 1)n^{2H-2} \text{ as } n \rightarrow \infty.$$

Thus, if  $H > \frac{1}{2}$ , then  $\sum_n \rho_H(n) = \infty$  and according to Beran's definition [2], fBm is called a long-memory process. If  $H < \frac{1}{2}$ , then  $\sum_n \rho_H(n) < \infty$  and fBm is called a short-memory process.

The long-memory property makes fBm as a potential candidate to model for noise in a variety of models (for a survey on theory and applications of long memory processes, see [9]). However, one has found that models driven by fBm are difficult to study because of the non-semimartingale property of fBm, as well as the complexity of the fractional stochastic calculus. In order to avoid this difficult, it would be desirable to find a long-memory process that has semimartingale property. The Proposition 4.1 below implies that  $W_t^{H,\varepsilon}$  is such a process.

**Proposition 4.1.**  *$W^{H,\varepsilon}$  has long-memory if  $H > \frac{1}{2}$  and has short-memory if  $H < \frac{1}{2}$ .*

*Proof.* Consider the auto-variance functions  $\rho_{H,\varepsilon}(n) := E(W_1^{H,\varepsilon}(W_{n+1}^{H,\varepsilon} - W_n^{H,\varepsilon}))$ ,  $n \geq 1$ . By Itô isometry formula we have

$$\begin{aligned} \rho_{H,\varepsilon}(n) &= \int_0^1 (1-s+\varepsilon)^\alpha [(n+1-s+\varepsilon)^\alpha - (n-s+\varepsilon)^\alpha] ds \\ &= \int_{\varepsilon-1}^\varepsilon (1+s)^\alpha [(n+1+s)^\alpha - (n+s)^\alpha] ds. \end{aligned}$$

We now apply the Mean Value Theorem and then Lagrange's theorem to obtain

$$\begin{aligned}\rho_{H,\varepsilon}(n) &= (1+s_0)^\alpha[(n+1+s_0)^\alpha - (n+s_0)^\alpha] \\ &= \alpha(1+s_0)^\alpha(s_0+\theta)^{\alpha-1},\end{aligned}$$

where  $s_0 \in (\varepsilon - 1, \varepsilon)$  and  $\theta \in (n, n + 1)$ .

Consequently,

$$\rho_{H,\varepsilon}(n) \approx \alpha(1+s_0)^\alpha n^{\alpha-1} \text{ as } n \rightarrow \infty,$$

which implies that  $\sum_n \rho_{H,\varepsilon}(n) = \infty$  if  $\alpha = H - \frac{1}{2} > 0$  and  $\sum_n \rho_{H,\varepsilon}(n) < \infty$  if  $\alpha < 0$ .

The Proposition thus is proved. □

Since  $W_t^{H,\varepsilon}$  is a semimartingale and, similar to fBm, has long memory, we suggest the name "fractional semimartingale" for  $W_t^{H,\varepsilon}$ .

**Definition 4.1.** The stochastic process  $W_t^{H,\varepsilon}$  defined by

$$W_t^{H,\varepsilon} = \int_0^t (t-s+\varepsilon)^\alpha dW_s, \alpha = H - \frac{1}{2}$$

is called a fractional semimartingale with two parameters  $(H, \varepsilon) \in (0, 1) \times (0, \infty)$ .

A fractional semimartingale  $W_t^{H,\varepsilon}$ , contrarily to fBm, is not a self-similar process and has non-stationary increments. However, a simulation example in Plienpanich et al. [26] has showed a significant reduction of error in a stock price model driven by  $W_t^{H,\varepsilon}$  as compared to the classical stock price model. This result means that  $W_t^{H,\varepsilon}$ , from empirical point of view, seems to be the potential candidate to model for noise in mathematical finance. In the remaining of this paper, let us study two models driven by fractional semimartingale.

**4.1. The Black-Scholes model.** We consider the Black-Scholes model with fractional noise that contains a stock  $S_t$  and a bond  $B_t$ .

Bond price:

$$dB_t = rB_t dt; \quad B_0 = 1 \tag{4.1}$$

Stock price:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^H, \tag{4.2}$$

where  $S_0$  is a positive real number, the coefficients  $r, \mu, \sigma$  are assumed to be constants symbolizing the riskless interest rate, the drift of the stock and its volatility, respectively. Then, from Theorem 3.2 we have

$$S_t = S_0 \exp(\mu t + \sigma W_t^H), \quad B_t = e^{rt}.$$

It is obvious that  $S_t \in C^{H^-}[0, T]$  and this implies that the fractional stochastic integral can be understood as Riemann-Stieltjes integral. As a consequence, our fractional Black-Scholes model admits an arbitrage opportunity (see, for instance, [29]) and we cannot use the traditional method to find the price of a European call option. However, for the Black-Scholes model with fractional semimartingale noise we have a surprising result in the following.

**Theorem 4.1.** *Consider the fractional semimartingale Black-Scholes model containing a bond (4.1) and a stock  $S_t^\varepsilon$*

$$dS_t^\varepsilon = \mu S_t^\varepsilon dt + \sigma S_t^\varepsilon dW_t^{H, \varepsilon}, \quad S_0^\varepsilon = S_0. \quad (4.3)$$

*Then this model has no arbitrage and is complete. The European call option price at time  $t = 0$  is given by*

$$C_0(\varepsilon) = S_0 N(d_1) - e^{-rT} K N(d_2), \quad (4.4)$$

where where  $K$  is trike price at maturity time  $T$ ,  $d_1 = \frac{\ln \frac{S_0}{K} + (r + \frac{\sigma^2 \varepsilon^{2\alpha}}{2})T}{\sigma \varepsilon^\alpha \sqrt{T}}$ ,  $d_2 = \frac{\ln \frac{S_0}{K} + (r - \frac{\sigma^2 \varepsilon^{2\alpha}}{2})T}{\sigma \varepsilon^\alpha \sqrt{T}}$  and  $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt$ .

*Proof.* Refer to [12, Theorem 4.1 and Remark 4.1]. □

**4.2. Optimal portfolio.** Let  $(\Omega, \mathcal{F}, P, \mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\})$  be a complete filtered probability space. Suppose that we observe continuously a single security with the price process  $\{S_t; 0 \leq t \leq T\}$  which follows the equation

$$dS_t = S_t(\mu_t dt + \sigma dW_t^{H_1, \varepsilon_1}), \quad (4.5)$$

where  $\sigma$  and initial price  $S_0$  are known positive constants, but the drift coefficient is an unobserved mean-reverting process with the dynamics

$$d\mu_t = \beta \mu_t dt + \nu dW_t^{H_2, \varepsilon_2}, \quad (4.6)$$

where  $\beta, \nu$  are unknown real constants, the initial condition  $\mu_0$  is a Gaussian random variable and independent of  $(W^{H_1, \varepsilon_1}, W^{H_2, \varepsilon_2})$ .

We assume the Brownian motions  $W^{(i)}, i = 1, 2$  are independent, so do  $W^{H_i, \varepsilon_i}, i = 1, 2$ . We shall denote by  $\mathcal{F}^S = \{\mathcal{F}_t^S, 0 \leq t \leq T\}$  the

$P$ -augmentation of the filtration generated by the price process  $S$ . A portfolio is an  $\mathbb{R}$ -valued  $\mathcal{F}^S$ -adapted process  $\pi = \{\pi_t, 0 \leq t \leq T\}$  such that

$$\int_0^T \pi_t^2 dt < \infty \text{ a.s.}$$

We regard  $\pi_t$  as the number of shares invested in the stock at time  $t$ . Given an initial wealth  $x \geq 0$ , the wealth process corresponding to a self-financing portfolio  $\pi$  is defined by  $X_0^{x,\pi} = x$  and

$$dX_t^{x,\pi} = \pi_t dS_t.$$

We denote by  $\mathcal{A}(x)$  the set of admissible portfolios  $\pi$  such that

$$X_t^{x,\pi} \geq 0 \text{ a.s., } 0 \leq t \leq T.$$

and consider the following expected logarithmic utility maximization problem from terminal wealth over the class  $\mathcal{A}(x)$  :

$$\max_{\pi \in \mathcal{A}(x)} E[\log(X_T^{x,\pi})]. \quad (4.7)$$

**Theorem 4.2.** *There exists an optimal portfolio  $\pi^* = \{\pi_t^*, 0 \leq t \leq T\}$  for the utility maximization problem (4.7):*

$$\pi_t^* = \frac{x(\hat{\mu}_t + \sigma\hat{\varphi}_t^{(1)})}{S_t \hat{L}_t}, \quad 0 \leq t \leq T, \quad (4.8)$$

where

$$\hat{L}_t = \exp\left(-\frac{1}{\sigma^2} \int_0^t (\hat{\mu}_s + \sigma\hat{\varphi}_s^{(1)}) dI_s - \frac{1}{2\sigma^2} \int_0^t (\hat{\mu}_s + \sigma\hat{\varphi}_s^{(1)})^2 ds\right),$$

where  $\hat{\mu}_s = E[\mu_s | \mathcal{F}_s^S]$ ,  $\hat{\varphi}_s^{(1)} = E[\varphi_s^{(1)} | \mathcal{F}_s^S]$ .

The optimal wealth process corresponding to  $\pi^*$  is given by

$$X_T^{x,\pi^*} = \frac{x}{\hat{L}_T}, \quad X_t^{x,\pi^*} = \frac{\pi_t^* S_t}{\hat{\mu}_t + \sigma\hat{\varphi}_t^{(1)}}, \quad 0 \leq t \leq T. \quad (4.9)$$

*Proof.* From the decomposition (2.2), Eq.(4.5) can be rewritten as follows

$$dS_t = S_t(\mu_t + \sigma\varphi_t^{(1)})dt + \sigma S_t \varepsilon_1^{H_1 - \frac{1}{2}} dW_t^{(1)},$$

where  $\varphi_t^{(1)} = \int_0^t (H_1 - \frac{1}{2})(t + \varepsilon_1 - u)^{H_1 - \frac{3}{2}} dW_u$  and we can use usual Itô calculus to find its solution

$$S_t = S_0 \exp \left( \int_0^t (\mu_s - \frac{1}{2} \sigma^2 \varepsilon_1^{2H_1 - 1}) ds + \sigma W_t^{H_1, \varepsilon_1} \right). \quad (4.10)$$

Denote by  $U = \{U_t, 0 \leq t \leq T\}$  the stochastic process defined by

$$U_t = \int_0^t \mu_s ds + \sigma W_t^{H_1, \varepsilon_1} = \sigma W_t^{(1)} + \int_0^t (\mu_s + \sigma \varphi_s^{(1)}) ds \quad (4.11)$$

then  $\mathcal{F}^S = \mathcal{F}^U$  and we can consider  $U$  as the observation process standing for  $S$ .

As in [19, Theorem 7.16] we define the innovation process  $I = \{I_t, 0 \leq t \leq T\}$

$$I_t = \frac{1}{\sigma} \left[ U_t - \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)}) ds \right].$$

$I$  is a  $\mathcal{F}^U$ -standard Brownian motion satisfying  $\mathcal{F}^I = \mathcal{F}^U$ .

We can see that the following stochastic process is a  $\mathcal{F}$ -martingale

$$\begin{aligned} L_t &:= \exp \left( -\frac{1}{\sigma} \int_0^t (\mu_s + \sigma \varphi_s^{(1)}) dW_s^{(1)} - \frac{1}{2\sigma^2} \int_0^t (\mu_s + \sigma \varphi_s^{(1)})^2 ds \right) \\ &= \exp \left( -\frac{1}{\sigma^2} \int_0^t (\mu_s + \sigma \varphi_s^{(1)}) dU_s + \frac{1}{2\sigma^2} \int_0^t (\mu_s + \sigma \varphi_s^{(1)})^2 ds \right). \end{aligned} \quad (4.12)$$

We take the conditional expectation on both two sides of (4.12) with respect to  $\mathcal{F}^U$  and obtain

$$\begin{aligned} \hat{L}_t &= \exp \left( -\frac{1}{\sigma^2} \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)}) dU_s + \frac{1}{2\sigma^2} \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)})^2 ds \right) \\ &= \exp \left( -\frac{1}{\sigma^2} \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)}) dI_s - \frac{1}{2\sigma^2} \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)})^2 ds \right). \end{aligned} \quad (4.13)$$

Applying the Itô formula we can see that  $\hat{L}_t X_t^{x,\pi}$  is a local martingale. Moreover, for  $x > 0$  and  $\pi \in \mathcal{A}(x)$  we have  $X_t^{x,\pi} \geq 0$ , whence  $\hat{L}_t X_t^{x,\pi}$  is an  $\mathcal{F}^U$ -supermartingale.

Now using a simple inequality that  $\log(u) - uv \leq \log \frac{1}{v} - 1, \forall u, v > 0$  we obtain the inequality below for all  $y > 0$

$$\begin{aligned} E[\log(X_T^{x,\pi})] &\leq E[\log(X_T^{x,\pi}) - y\hat{L}_T X_T^{x,\pi}] + yx \\ &\leq E\left[\log\left(\frac{1}{y\hat{L}_T}\right) - 1\right] + yx. \end{aligned} \quad (4.14)$$

The equalities in (4.14) hold if and only if

$$X_T^{x,\pi} = \frac{x}{\hat{L}_T} \text{ a.s.} \quad (4.15)$$

Thus the strategy  $\pi$  satisfying (4.15) is optimal.

From (4.13) we have

$$d\frac{x}{\hat{L}_t} = \frac{x(\hat{\mu}_s + \sigma\hat{\varphi}_s^{(1)})}{\hat{L}_t} dY_t = \frac{x(\hat{\mu}_s + \sigma\hat{\varphi}_s^{(1)})}{S_t\hat{L}_t} dS_t,$$

hence, if we put

$$\pi_t^* = \frac{x(\hat{\mu}_t + \sigma\hat{\varphi}_t^{(1)})}{S_t\hat{L}_t}, 0 \leq t \leq T,$$

then  $dX_t^{x,\pi^*} = \pi_t^* dS_t = d\frac{x}{\hat{L}_t}$ . This yields

$$X_t^{x,\pi^*} = \frac{x}{\hat{L}_t}, 0 \leq t \leq T$$

because  $X_0^{x,\pi^*} = x = \frac{x}{\hat{L}_0}$ .

In particular,  $\pi^*$  satisfies (4.15) and it is the desired optimal strategy.

The proof is thus complete.  $\square$

At the end of this paper, we put  $Z_t = (\mu_t, \varphi_s^{(1)})$  and our aim is to compute the optimal filter  $\hat{Z}_t = (\hat{\mu}_t, \hat{\varphi}_s^{(1)})$  appeared in Theorem 4.2. First, in our filtering problem the dynamics of observation  $U$  and state process  $\mu$  are given by, respectively

$$\begin{aligned} dU_t &= \mu_t dt + \sigma dW_t^{H_1, \varepsilon_1}, \\ d\mu_t &= \beta\mu_t dt + \nu dW_t^{H_2, \varepsilon_2}. \end{aligned}$$

The error matrix  $P(t, s)$  is defined by

$$P(t, s) = E[Z_t(Z_s - \hat{Z}_s)^\tau], \quad 0 \leq s \leq t \leq T.$$

**Theorem 4.3.** *The optimal filter  $\hat{Z}_t = (\hat{\mu}_t, \hat{\varphi}_t^{(1)})$  satisfies the stochastic integral equation*

$$\hat{Z}_t = \frac{1}{\sigma^2} \int_0^t [P(t, s) + D(t, s)] a [dU_s - a^\tau \hat{Z}_s ds], \quad 0 \leq t \leq T, \quad (4.16)$$

where  $a = (1, \sigma)^\tau$ ,  $\tau$  denotes the transposition and

$$D(t, s) = \begin{pmatrix} 0 & 0 \\ \sigma(H_1 - \frac{1}{2})(t - s + \varepsilon_1)^{H_1 - \frac{3}{2}} & 0 \end{pmatrix}.$$

The error matrix  $P(t, s)$  is the solution of the following Riccati-type equation

$$P(t, s) = \frac{-1}{\sigma^2} \int_0^s [P(t, u) + D(t, u)] a a^\tau [P(s, u) + D(s, u)]^\tau du + \Gamma_{ZZ}(t, s), \quad 0 \leq s \leq t \leq T, \quad (4.17)$$

where  $\Gamma_{ZZ}(t, s) = E[Z_t Z_s^\tau]$ .

*Proof.* We have

$$U_t = \int_0^t (\mu_s + \sigma \varphi_s^{(1)}) ds + \sigma W_t^{(1)}$$

and the associated innovation process  $\{I_t, 0 \leq t \leq T\}$  is given by

$$I_t = \frac{1}{\sigma} \left[ U_t - \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)}) ds \right], \quad (4.18)$$

and is a  $\mathcal{F}^U$ -standard Brownian motion satisfying  $\mathcal{F}^I = \mathcal{F}^U$ . For convenience, we rewrite  $I_t$  in the matrix form:

$$I_t = \frac{1}{\sigma} \int_0^t (Z_s - \hat{Z}_s)^\tau a ds + W_t^{(1)}.$$

Since the Brownian motions  $W^{(1)}, W^{(2)}$  are independent, the system  $(Z, U) = (\mu, \varphi^{(1)}, U)$  is Gaussian. Hence, the optimal filter  $\hat{Z}$  is a linear function of the observation process  $\{U_s, 0 \leq s \leq t\}$  and it is also a Gaussian system. By [19, Theorem 5.6], there exists a deterministic



Volterra function  $F(t, s) = (F_1(t, s), F_2(t, s))$  on  $0 \leq s \leq t \leq T$  such that

$$\int_0^t |F(t, s)|^2 ds < \infty, \quad 0 \leq t \leq T,$$

$$\hat{Z}_t = \int_0^t F(t, s) dI_s, \quad 0 \leq t \leq T. \quad (4.19)$$

Now from (4.19) we can find the function  $F(t, s)$  as follows

$$\begin{aligned} F(t, s) &= \frac{d}{ds} E(\hat{Z}_t I_s) = \frac{d}{ds} E(Z_t I_s) \\ &= \frac{d}{ds} \left( \frac{1}{\sigma} \int_0^s E[Z_t (Z_u - \hat{Z}_u)^\tau] a du + E[Z_t W_s^{(1)}] \right) \\ &= \frac{1}{\sigma} P(t, s) a + (0, (H_1 - \frac{1}{2})(t - s + \varepsilon_1)^{H_1 - \frac{3}{2}})^\tau \\ &= \frac{1}{\sigma} [P(t, s) + D(t, s)] a. \end{aligned}$$

Thus we have

$$\hat{Z}_t = \frac{1}{\sigma} \int_0^t [P(t, s) + D(t, s)] a dI_s, \quad 0 \leq t \leq T, \quad (4.20)$$

and the filtering equation (4.16) follows from (4.18).

From the definition of the error matrix we have

$$P(t, s) = E[Z_t Z_s^\tau] - E[\hat{Z}_t \hat{Z}_s^\tau],$$

hence the equation (4.17) follows from (4.20) and Itô isometry formula.

We give an explicit computation of  $\Gamma_{ZZ}(t, s), 0 \leq s \leq t \leq T$ . We have

$$\Gamma_{ZZ}(t, s) = \begin{pmatrix} E[\mu_t \mu_s] & E[\mu_t \varphi_s^{(1)}] \\ E[\varphi_t^{(1)} \mu_s] & E[\varphi_t^{(1)} \varphi_s^{(1)}] \end{pmatrix}.$$

We recall from [14, Proposition 4.2] that

$$\mu_t = \mu_0 e^{\beta t} + \nu \varepsilon_2^{H_2 - \frac{1}{2}} \int_0^t e^{\beta(t-s)} dW_s^{(2)} + \nu \int_0^t e^{\beta(t-s)} \varphi_s^{(2)} ds.$$

Applying stochastic Fubini's theorem we have

$$\begin{aligned} \int_0^t e^{\beta(t-s)} \varphi_s^{(2)} ds &= \int_0^t \int_0^s (H_2 - \frac{1}{2}) e^{\beta(t-s)} (s - u + \varepsilon_2)^{H_2 - \frac{3}{2}} dW_u^{(2)} ds \\ &= \int_0^t \int_u^t (H_2 - \frac{1}{2}) e^{\beta(t-s)} (s - u + \varepsilon_2)^{H_2 - \frac{3}{2}} ds dW_u^{(2)}. \end{aligned}$$

Thus,

$$\mu_t = \mu_0 e^{\beta t} + \int_0^t b(t-u) dW_u^{(2)},$$

where

$$\begin{aligned} b(t-u) &= \nu \varepsilon_2^{H_2 - \frac{1}{2}} e^{\beta(t-u)} + \nu \int_u^t (H_2 - \frac{1}{2}) e^{\beta(t-s)} (s - u + \varepsilon_2)^{H_2 - \frac{3}{2}} ds \\ &= e^{\beta(t-u)} (\nu \varepsilon_2^{H_2 - \frac{1}{2}} + \nu \int_0^{t-u} (H_2 - \frac{1}{2}) e^{-\beta s} (s + \varepsilon_2)^{H_2 - \frac{3}{2}} ds). \end{aligned}$$

As a consequence, we have

$$\begin{aligned} E[\mu_t \mu_s] &= e^{\beta(t+s)} E[\mu_0^2] + \int_0^s b(t-u) b(s-u) du, \\ E[\mu_t \varphi_s^{(1)}] &= E[\varphi_t^{(1)} \mu_s] = 0, \\ E[\varphi_t^{(1)} \varphi_s^{(1)}] &= (H_1 - \frac{1}{2})^2 \int_0^s (t-u + \varepsilon_1)^{H_1 - \frac{3}{2}} (s-u + \varepsilon_1)^{H_1 - \frac{3}{2}} du. \end{aligned}$$

□

When  $H_1 = H_2 = \frac{1}{2}$  and  $\mu_0 = 0$  we obtain the following corollary for the classical Kalman-Bucy linear filtering problem.

**Corollary 4.1.** *Consider the filtering problem with observation and state process given by, respectively*

$$\begin{aligned} dU_t &= \mu_t dt + \sigma dW_t^{(1)}, \quad U_0 = 0, \\ d\mu_t &= \beta \mu_t dt + \nu dW_t^{(2)}, \quad \mu_0 = 0. \end{aligned}$$

Then the optimal filter  $\hat{\mu}_t$  satisfies

$$\hat{\mu}_t = \frac{\beta - \gamma(t)}{\sigma^2} \hat{\mu}_t dt + \frac{\gamma(t)}{\sigma^2} dU_t,$$

where the error  $\gamma(t) = E[\mu_t(\mu_t - \hat{\mu}_t)]$  is the solution of the Riccati equation

$$\frac{d\gamma(t)}{dt} = \frac{\nu^2}{\sigma^2} + \frac{2\beta}{\sigma^2} \gamma(t) - \frac{1}{\sigma^2} \gamma^2(t).$$

*Proof.* It is obvious since  $D(t, s) \equiv 0 \equiv \varphi_t^{(1)}$ ,

$$\Gamma_{ZZ}(t, s) = \begin{pmatrix} \frac{\nu^2}{2\beta}(e^{\beta(t+s)} - e^{\beta(t-s)}) & 0 \\ 0 & 0 \end{pmatrix}.$$

$$P(t, s) = \begin{pmatrix} E[\mu_t(\mu_s - \hat{\mu}_s)] & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$E[\mu_t(\mu_s - \hat{\mu}_s)] = E[E[\mu_t | \mathcal{F}_s](\mu_s - \hat{\mu}_s)] = e^{\beta(t-s)} \gamma(s).$$

□

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## REFERENCES

- [1] E. Alòs, O. Mazet and D. Nualart. *Stochastic Calculus with Respect to Gaussian Processes*. The Annals of Probability, 2001, Vol. 29, No. 2, 766-801.
- [2] J. Beran and N. Terrin. Errata. *Testing for a change of the longmemory parameter*. Biometrika (1996), 83 (3): 627-638.
- [3] P. Carmona, L. Coutin, and G. Montseny, *Stochastic integration with respect to fractional Brownian motion*, Ann. Inst. H. Poincaré Probab. Statist., 39(1):27-68, 2003.
- [4] P. Cheridito, D. Nualart, *Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter H in (0, 1/2)*, Ann. Inst. H. Poincaré Probab. Statist. 41 (2005) 1049-1081.
- [5] F. Comte, E. Renault, *Long memory in continuous-time stochastic volatility models*, Math. Finance 8 (4) (October 1998) 291-323.
- [6] L. Coutin, *An Introduction to Stochastic Calculus with Respect to Fractional Brownian motion*. In *Séminaire de Probabilités XL*, page 3-65. Springer-Verlag Berlin Heidelberg, 2007.
- [7] L. Decreusefond, A.S. Üstünel, *Stochastic analysis of the fractional Brownian motion*, Potential Anal. 10 (1999) 177-214.
- [8] L. Decreusefond, *Stochastic integration with respect to Volterra processes*, Ann. Inst. H. Poincaré Probab. Statist., 41(2):123-149, 2005.

- [9] P. Doukhan, G. Oppenheim and M.S. Taqqu, Theory and applications of long-range dependence, *Birkhäuser Boston, Inc., Boston, MA, 2003*.
- [10] R.M. Dudley, *The sizes of compact subsets of Hilbert space and continuity of Gaussian processes*, J. Funct. Anal. 1, 290-330 (1967).
- [11] N.T. Dung, *A class of fractional stochastic differential equations*, Vietnam Journal of Mathematics 36: 3(2008) 271-279.
- [12] N.T. Dung, *Semimartingale approximation of Fractional Brownian motion and its applications*. Comput. Math. Appl. 61 (2011), no. 7, 1844-1854.
- [13] N.T. Dung, *Fractional geometric mean-reversion processes*, J. Math. Anal. Appl. 380 (2011), no. 1, 396-402.
- [14] N.T. Dung and T.H. Thao, *An Approximate Approach to Fractional Stochastic Integration and Its Applications*, Braz. J. Probab. Stat. 24 (1) (2010) 57-67.
- [15] Yu-J. Jien and J. Ma *Stochastic differential equations driven by fractional Brownian motions*, Bernoulli 15(3), 2009, 846-870.
- [16] J.A. Léon and J. San Martín, *Linear stochastic differential equations driven by a fractional Brownian motion with Hurst parameter less than 1/2*, Stoch. Anal. Appl. 25(2006) 105126.
- [17] S.C. Lim and V.M. Sithi. *Asymptotic properties of the fractional Brownian motion of Riemann-Liouville type*. Physics Letters A 206 (1995) 311-317.
- [18] S.J. Lin, *Stochastic analysis of fractional Brownian motions*, Stochastics and Stoch. Reports, 55(1995) 121-140.
- [19] R.S. Liptser and A.N. Shiryaev, *Statistics of Random Processes, I General Theory, Vol.5, Springer, New York, second edition, 2001*.
- [20] B. B. Mandelbrot and J. Van Ness, *Fractional Brownian motion, fractional noises and applications*, SIAM Review, 10 (1968), 422-437.
- [21] Y.S. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes. Springer, 2008*.
- [22] I. Nourdin, *Schémas d'approximation associés à une équation différentielle dirigée par une fonction höldérienne; cas du mouvement brownien fractionnaire*. C. R. Math. Acad. Sci. Paris, 340(8):611-614, 2005
- [23] I. Nourdin and C.A. Tudor, *Some linear fractional stochastic equations*, Stochastics 78(2006) 51-65.
- [24] D. Nualart and A. Răşcanu, *Differential equations driven by fractional Brownian motion*, Collectanea Mathematica. 53 (2002) 55-81.
- [25] D. Nualart, *The Malliavin Calculus and Related Topics. 2nd edition, Springer 2006*.
- [26] T. Plienpanich, P. Sattayatham, T.H. Thao, *Fractional integrated GARCH diffusion limit models*, J. Korean Statist. Soc. 38 (2009), no. 3, 231-238.
- [27] F. Russo, P. Vallois, *Forward, backward and symmetric stochastic integration*, Probab. Theory Related Fields 97(1993), 97-109.
- [28] F. Russo, P. Vallois, *Stochastic calculus with respect to a finite quadratic variation process*, Stoch. Stoch. Rep. 70(2000), 1-40.
- [29] A.N. Shiryaev, *On arbitrage and replication for fractal models*, Preprint, MaPhySto, Aarhus, 2001.
- [30] T.H. Thao, *An Approximate Approach to Fractional Analysis for Finance*, Nonlinear Analysis 7(2006), 124-132.
- [31] L.C. Young, *An inequality of the Hölder type connected with Stieltjes integration*, Acta Math. 67 (1936) 251-282.

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